ON THE GLOBAL WELL-POSEDNESS OF 2-D BOUSSINESQ SYSTEM WITH VARIABLE VISCOSITY

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ABSTRACT. In this paper, we investigate the global well-posedness of 2-D Boussinesq system, which has variable kinematic viscosity and with thermal conductivity of $|D|\theta$, with general initial data provided that the viscosity coefficient is sufficiently close to some positive constant in L^∞ norm.

Keywords: Boussinesq systems, Littlewood-Paley Theory, variable viscosity

AMS Subject Classification (2000): 35Q30, 76D03

1. Introduction

In this paper, we consider the global well-posedness to the following two-dimensional Boussinesq equations with variable viscous coefficient

$$
\left\{\begin{array}{ll}
\partial_t\theta + u\cdot\nabla\theta + \nu|D|\theta = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\
\partial_t u + u\cdot\nabla u - \operatorname{div}(2\mu(\theta)d) + \nabla\Pi = \theta e_2, & \\
\operatorname{div} u = 0, & \\
(\theta,u)|_{t=0} = (\theta_0,u_0), &
\end{array}\right.
\tag{1.1}
$$

where $\theta, u = (u_1,u_2)$ stand for the temperature and velocity of the fluid respectively, $d = \big(d_{ij}\big)_{2\times 2}$ with $d_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{i,j}$ denotes the deformation matrix, Π is a scalar pressure function, and in general, and the kinematic viscous coefficient $\mu(\theta)$ is a smooth, positive and non-decreasing function on $[0,\infty)$. The thermal conductivity coefficient $\nu\geq 0$, and $e_2=(0,1)$, θe_2 denotes buoyancy force. Furthermore, in all that follows, we shall always denote $|D|^s$ to be the Fourier multiplier with symbol $|\xi|^s$.

The Boussinesq system arises from a zeroth order approximation to the coupling between Navier-Stokes equations and the thermodynamic equations. It can be used as a model to describe many geophysical phenomena ([26]). In the Boussinesq approximation of a large class of flow problems, thermodynamic coefficients such as kinematic viscosity, specific heat and thermal conductivity may be assumed to be constants, leading to a coupled system of parabolic equations.

However, there are some fluids such as lubrificants or some plasma flow for which this is not an accurate assumption [19, 27], and a quasilinear parabolic system as follows has to be considered:

$$
\left\{\begin{array}{ll}
\partial_t\theta + u\cdot\nabla\theta - \Delta\varphi(\theta) = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\
\partial_t u + u\cdot\nabla u - \operatorname{div}(2\mu(\theta)d) + \nabla\Pi = F(\theta), & \\
\operatorname{div} u = 0. &
\end{array}\right.
\tag{1.2}
$$

One may check [18] and the references therein for more details about (1.2). Under some technical assumptions, the global existence of weak solutions to (1.2) and in the case of constant viscosity, the uniqueness of such weak solutions in two space dimension was proved in [18].

Recently big progresses have been made on the global well-posedness of the System (1.2) especially with $F(\theta)=\theta e_2$ in (1.2) and in two space dimension. In this case, Wang and Zhang [29] proved the global existence of smooth solutions to (1.2) under the assumptions that both $\varphi'(\cdot)$

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and $\mu(\cdot)$ belong to $L^\infty(\mathbb{R}^+)$ and have positive lower bounds. We remark that the most crucial part in [29] is to use De-Giorgi method to derive the *a priori* estimate of $\|\theta\|_{L^\infty((\delta,T);C^\alpha)}$ for some $\alpha > 0$ and any $\delta \in]0,T[$. Even with $\varphi(\theta) = 0$ and $\mu(\theta) = \mu > 0$ in (1.2), Chae [10] and Hou, Li [25] independently proved the global existence of smooth solutions to (1.2), the first author of this paper and Hmidi [3] established the global well-posedness of this system with initial data satisfying $(\theta_0, u_0) \in B^0_{2,1}(\mathbb{R}^2) \times \big(L^2 \cap B^{-1}_{\infty,1}\big)(\mathbb{R}^2)$. When $\varphi'(\theta) = \nu > 0$ and $\mu(\theta) = 0$, Hmidi and Keraani [21] proved the global existence and uniqueness of solutions to (1.2) with $u_0 \in H^s(\mathbb{R}^2)$ and $\theta_0 \in B^0_{2,1}(\mathbb{R}^2) \cap B^0_{p,\infty}(\mathbb{R}^2)$ for $s \in]0,2]$ and $p \in]2,\infty]$. We also mention the 3-D well-posedness result on this problem with small initial data in [15]

There are also many studies on the so-called Boussinesq system with critical dissipation in two space dimension, which reads

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \nu|D|\theta = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ \partial_t u + u\cdot\nabla u + \mu|D|u + \nabla\Pi = \theta e_2,\\ \operatorname{div} u = 0,\\ (\theta,u)|_{t=0} = (\theta_0,u_0). \end{cases} \tag{1.3}$$

When $\nu = 0$ and $\mu > 0$, the above system is called Boussinesq-Navier-Stokes system with critical dissipation, Hmidi, Keraani and Rousset [22] proved the global well-posedness of such system. When $\nu > 0$ and $\mu = 0$, the System (1.3) is called Boussinesq-Euler system with critical dissipation, Hmidi, Keraani and Rousset [23] proved its global well-posedness. We emphasize that the authors of [22, 23] used crucially the structure of the System (1.3), namely, the quantity, $\Gamma = \omega + R\theta$, for $\omega = \partial_1 u_2 - \partial_2 u_1$ and the Riesz transform $R = \partial_1/|D|$. Very recently even the logarithmically critical Boussinesq system was investigated by Hmidi in [20]. There are also studies to the global well-posedness of the anisotropic Boussinesq system (with partial thermal conductivity and partial kinematic viscosity) in two space dimension (see [9, 16] for instance).

On the other hand, the first author of this paper [1] proved the global well-posedness of (1.2) in two space dimension under the assumptions that: $\varphi(\theta) = 0$, $F(\theta) = 0$, and the initial data satisfies $\theta_0 \in \dot B^1_{2,1}(\mathbb{R}^2)$, $u_0 \in \big(L^2 \cap \dot B^{-1}_{\infty,1}\big)(\mathbb{R}^2)$, moreover for some sufficiently small ε, there holds

$$\|\theta_0\|_{\dot B^1_{2,1}} + \|\mu(\theta_0) - 1\|_{L^\infty} \le \varepsilon.$$

Motivated by [1] and the recent results of the authors [4, 5] concerning the global well-posedness of inhomogeneous Navier-Stokes system with variable density (see also [17, 24]), we [6] proved the global well-posedness of (1.2) in 3-D with $\varphi = 0$, $F(\theta) = 0$ and with initial data $\theta_0 \in \big(B^1_{3,1} \cap B^{(1/2)+}_{\infty,\infty}\big)(\mathbb{R}^3)$ and $u_0 \in \big(\dot H^{-2\delta} \cap \dot B^0_{3,1}\big)(\mathbb{R}^3)$ for some $\delta \in]0,1/2[$, provided that there exist a sufficiently small constant ε_0, and some small enough constant ε, which depends on $\|\theta_0\|_{\dot B^1_{3,1}\cap B^{(1/2)+}_{\infty,\infty}}$, such that

$$\|\mu(\theta_0) - 1\|_{L^\infty} \le \varepsilon_0 \quad \text{and} \quad \|u_0\|_{\dot B^0_{3,1}} \le \varepsilon.$$

The purpose of this paper is to prove the global well-posed of the System (1.1) with general initial data and under the assumption of (1.5).

In what follows, we shall always make the following conventions that $\nu = 1$ in (1.1), and

$$0 < 1 \le \mu(\theta), \quad \mu(\cdot) \in W^{2,\infty}(\mathbb{R}^+), \quad \mu(0) = 1, \quad \text{and} \quad p^* \stackrel{\text{def}}{=} 1\Big/C\|1-\mu(\cdot)\|_{L^\infty} \tag{1.4}$$

for some large enough positive constant C.

The main result of this paper states as follows:

Theorem 1.1. *Let $q \in]1,4/3[$, $p \in \big]4,p^*\big]$ and $s_0 \in \big]1, 2(2/q-1)\big[$. Let $\theta_0 \in \big(L^q \cap \dot H^{-s_0} \cap H^{1/2}\big) \cap B^{1/2}_{p,\infty}$ and $u_0 \in \dot B^{-1}_{\infty,1} \cap H^1$ be a solenoidal vector filed. Then there exists some sufficiently small ε_0 so that if we assume*

$$\|\mu(\cdot) - 1\|_{L^\infty(\mathbb{R}^+)} \le \varepsilon_0, \tag{1.5}$$

(1.1) *has a unique global solution* (θ, u) *so that*

$$\begin{aligned}\theta \in \mathcal{C}\big([0,\infty); L^q\cap \dot H^{-s_0}\cap H^{1/2}\big)\cap L^\infty(\mathbb{R}^+; B^{1/2}_{p,\infty})\\ \cap L^2(\mathbb{R}^+;H^1)\cap \widetilde L^1_{loc}(\mathbb{R}^+;B^{3/2}_{p,\infty}) \quad \text{and}\\ u\in \mathcal{C}([0,\infty);H^1)\cap \widetilde L^2(\mathbb{R}^+;\dot B^{3/2}_{2,\infty})\cap L^1_{loc}(\mathbb{R}^+;\dot B^1_{\infty,1}),\quad \partial_t u\in L^2(\mathbb{R}^+;L^2).\end{aligned} \tag{1.6}$$

Furthermore, there holds

$$\|\theta(t)\|_{L^2}\le CE_0\langle t\rangle^{-s_0} \quad \text{for } \langle t\rangle \stackrel{\text{def}}{=} e+t, \tag{1.7}$$

where

$$E_0\stackrel{\text{def}}{=}\mathcal{E}_0\big(1+\mathcal{E}_0\big) \quad \text{and} \quad \mathcal{E}_0\stackrel{\text{def}}{=}\|\theta_0\|_{L^2\cap\dot H^{-s_0}}+\|u_0\|_{L^2}+\|\theta_0\|_{L^q}\big(\|u_0\|_{L^2}+\|\theta_0\|_{L^q}\big). \tag{1.8}$$

We shall present the functional space framework in Section 3

Remark 1.1. *Let us give the following remarks concerning this theorem:*

1. *The above theorem works for viscous coefficient of the type* $\mu(\theta)=1+\varepsilon_0\zeta(\theta)$ *with* $\zeta\in L^\infty$.
2. *Note from the proof of Proposition 4.1 of* [29] *that: for a smooth enough solution* (θ,u) *of* (1.2), *the a priori estimates of* $\|u\|_{L^\infty_t(L^2)\cap L^2_t(\dot H^1)}$ *and* $\|\theta\|_{L^\infty_t(L^2)\cap L^2_t(\dot H^1)}$, *which can be provided by energy conservation law, are almost critical to use De-Giorgi method to derive the space Hölder estimate for* θ. *With the a priori estimates of* $\|u\|_{L^\infty_t(L^2)\cap L^2_t(\dot H^1)}$ *and* $\|\theta\|_{L^\infty_t(L^2)\cap L^2_t(\dot H^{1/2})}$, *we do not know how to go through the proof of Proposition 4.1 of* [29], *which is the case for the System* (1.1).
3. *It is easy to observe that for the System* (1.1) *with variable viscosity* $\mu(\theta)$, *the quantity,* $\Gamma=\omega+R\theta$, *does not satisfy a "good" equation as that in in* [22, 23]. *Hence it is not clear to use the approach in* [22, 23] *to deal with the well-posedness theory of* (1.1).
4. *Compared with the results in* [4, 5] *and* [24] *for the inhomogeneous Navier-Stokes system with variable viscosity, here* ε_0 *is a uniform small positive constant, which does not depend on* θ_0. *While in* [4, 5] *and* [24], *the smallness condition for* $\mu(\rho_0)-1$ *is in some sense formulated as*
$$\|\mu(\rho_0)-1\|_{L^\infty}\big(1+\|\rho_0\|_{B^\delta_{\infty,\infty}}\big)\le\varepsilon_0$$
for some $\delta>0$. *In general, under the assumption that*
$$\|\mu(\rho_0)-1\|_{L^\infty}\le\varepsilon_0 \tag{1.9}$$
for some ε_0 *sufficiently small, Desjardins* [17] *only proved the global existence of strong solutions for 2-D inhomogeneous Navier-Stokes system. Yet the uniqueness and regularities of such strong solutions are still open.*
5. *Let us emphasize that the idea of the derivation of pseudo-energy conservation introduced by Desjardins in* [17] *also play a key role in the proof of Theorem 1.1. However, due to the appearance of the diffusion term* $|D|\theta$ *in the* θ *equation of* (1.1), *the argument is much more complicated here, which we shall explain in Section 2 for more details. Moreover, the basic energy law (see* (A.4) *in the Appendix) to the System* (1.1) *grows like* e^t *as time* t *goes to* ∞. *To overcome this difficulty, we made some technical assumption for the low frequency part of* θ_0 *in Theorem 1.1 so that there holds* (1.7).

We also have the following corollary:

Corollary 1.1. *Under the assumptions of Theorem 1.1, we can replace the assumption* (1.5) *by requiring that*

$$\big(\mu(\cdot)-1\big)^2 \quad \text{is a convex function of its variable} \quad \text{and} \quad \|\mu(\theta_0)-1\|_{L^\infty}\le\varepsilon_0, \tag{1.10}$$

then (1.1) *still has a unique global solution* (θ,u) *which satisfy* (1.6), (1.7) *and* (1.8).

We remark that (1.10) works in particular for $\mu(\theta)=1+\theta$, which is a physical shallow water type viscosity, with $\|\theta_0\|_{L^\infty}\leq\varepsilon_0$.

Let us complete this section with the notations we are going to use in this context.

Notations: Let A,B be two operators, we denote $[A;B]=AB-BA$, the commutator between A and B. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leq Cb$. We shall denote by $(a|b)$ (or $(a|b)_{L^2}$) the $L^2(\mathbb{R}^2)$ inner product of a and b, and denote by $(d_j)_{j\in\mathbb{Z}}$ (resp. $(c_j)_{j\in\mathbb{Z}}$) a generic element of $\ell^1(\mathbb{Z})$ (resp. $\ell^2(\mathbb{Z})$) so that $\big\|(d_j)_{j\in\mathbb{Z}}\big\|_{\ell^1(\mathbb{Z})}=1$ (resp. $\big\|(c_j)_{j\in\mathbb{Z}}\big\|_{\ell^2(\mathbb{Z})}=1$).

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;\,X)$ the set of continuous functions on I with values in X. For $q\in[1,+\infty]$, the notation $L^q(I;\,X)$ stands for the set of measurable functions on I with values in X, such that $t\longmapsto\|f(t)\|_X$ belongs to $L^q(I)$. Finally for any vector field $v=(v_1,v_2)$, we denote $d(v)=\frac{1}{2}\big(\partial_iv_j+\partial_jv_i\big)_{i,j=1,2}$, and the Leray projection operator $\mathbb{P}\stackrel{\text{def}}{=}Id+\nabla(-\Delta)^{-1}\text{div}$.

2. Strategies to the proof of Theorems 1.1

As the existence part of Theorems 1.1 basically follows from the *a priori* estimates for smooth enough solutions of (1.1). We shall only outline the main steps in the derivation of the *a priori* estimates.

The first step to prove Theorem 1.1 is to use Schonbek's strategy in [28] (see also [30]) together with the energy law of the System (1.1) to prove that under the assumptions of Theorem 1.1, there holds (1.7) and

$$\|u\|_{L^\infty_t(L^2)}+\|\nabla u\|_{L^2_t(L^2)}\leq CE_0 \tag{2.1}$$

for E_0 given by (1.8).

While for any time interval $I=[I^-,I^+]$ of $\mathbb{R}^+$, it follows, by a similar derivation of the conservation of pseudo-energy for 2-D inhomogeneous Navier-Stokes system in [17], that

$$\|\nabla u\|^2_{L^\infty(I;L^2)}+\|\partial_tu\|^2_{L^2(I;L^2)}\leq C\big(1+\|\nabla u(I^-)\|^2_{L^2}\big)\exp\big(CE_0^2(1+E_0^2)\big)\exp\Big(C\|\nabla\theta\|^2_{L^2(I;L^2)}\Big). \tag{2.2}$$

In order to control the estimate of $\|\nabla\theta\|_{L^2(I;L^2)}$ on the right-hand side of (2.2), we get, by using $\dot H^{1/2}$ energy estimate to the θ equation of (1.1), that

$$\begin{aligned}\|\theta\|^2_{\widetilde L^\infty(I;\dot H^{1/2})}+\|\theta\|^2_{L^2(I;\dot H^1)}\leq&\|\theta(I^-)\|^2_{\dot H^{1/2}}+C\|\nabla u\|^2_{L^2(I;L^2)}\Big(1+\|\theta_0\|^2_{L^2\cap L^\infty}\\&+\|\theta_0\|^2_{L^\infty}\ln\big(1+\|\theta(I^-)\|_{B^{1/2}_{p,\infty}}+\|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2(I;L^p)}\big)\Big).\end{aligned} \tag{2.3}$$

On the other hand, we deduce from $\operatorname{div}u=0$ that

$$\nabla u=\nabla(-\Delta)^{-1}\text{div}\mathbb{P}\big(2(\mu(\theta)-1)d\big)-\nabla(-\Delta)^{-1}\text{div}\mathbb{P}\big(2\mu(\theta)d\big), \tag{2.4}$$

from which and (1.5), we achieve

$$\|\nabla u\|_{L^2(I;L^p)}\leq CE_0(1+E_0)\big(1+\|\nabla u\|_{L^\infty(I;L^2)}+\|\partial_tu\|_{L^2(I;L^2)}\big)\quad\forall p\in[4,p^*]. \tag{2.5}$$

Substituting (2.3) and (2.5) into (2.2) gives rise to

$$\begin{aligned}\|\nabla u\|^2_{L^\infty(I;L^2)}+\|\partial_tu\|^2_{L^2(I;L^2)}\leq CA(I^-)\Big(&1+\|\theta(I^-)\|_{B^{1/2}_{p,\infty}}\\&+E_0(1+E_0)\|\theta_0\|_{L^\infty}\big(1+\|\nabla u\|_{L^\infty(I;L^2)}+\|\partial_tu\|_{L^2(I;L^2)}\big)\Big)^{C\|\theta_0\|^2_{L^\infty}\|\nabla u\|^2_{L^2(I;L^2)}},\end{aligned} \tag{2.6}$$

for $A(I^-)$ given by (4.42). Therefore to close the estimate in (2.6), we need to prove that there is no energy concentration on any small time interval I, which is the purpose of Lemma 4.4.

With (2.6) and Lemma 4.4 in hand, one can deduce the global in time estimate of $\|\nabla u\|_{L^\infty_t(L^2)}$ and $\|\nabla u\|_{L^2_t(L^p)}$ for $p \in [4, p^*]$ by a boot-strap argument. At this stage, by using Littlewood-Paley theory, we can derive the estimate of $\|u\|_{L^1_t(\dot{B}^1_{\infty,1})}$ for any $t < \infty$.

With all these *a priori* estimates obtained in Section 4, we complete the existence part of Theorem 1.1 in Section 5 by smoothing the initial data and using a standard Lions-Aubin's Lemma argument. Finally the uniqueness part of Theorem 1.1 will be proved by an Osgood Lemma argument.

3. Littlewood-Paley analysis and preliminary estimates

3.1. **Basic facts on Littlewood-Paley theory.** The proof of Theorem 1.1 requires Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x \in \mathbb{R}^2$ (see e.g. [7]). Let φ be a smooth function supported in the annulus $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2, 3/4 \le |\xi| \le 8/3\}$ and $\chi(\xi)$ be a smooth function supported in the ball $\mathcal{B} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2,\ |\xi| \le 4/3\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \neq 0 \quad \text{and} \quad \chi(\xi) + \sum_{q\ge 0} \varphi(2^{-q}\xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^2 .$$

Then for $u \in \mathcal{S}'_h$ (see Definition 1.26 of [7]), which means $u \in \mathcal{S}'$ and $\lim_{j\to-\infty} \|\chi(2^{-j}D)u\|_{L^\infty} = 0$, we set

$$\begin{aligned}
&\forall\, j \in \mathbb{Z}, \quad \dot{\Delta}_j u \stackrel{\text{def}}{=} \varphi(2^{-j}\mathrm{D})u \quad \text{and} \quad \dot{S}_j u \stackrel{\text{def}}{=} \chi(2^{-j}D)u,\\
&\forall\, q \ge 0, \quad \Delta_q u \stackrel{\text{def}}{=} \varphi(2^{-q}\mathrm{D})u, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \chi(D)u \quad \text{and}\\
&\forall\, q \le -2, \quad \Delta_q u = 0, \quad S_q u \stackrel{\text{def}}{=} \sum_{-1\le q'\le q-1} \Delta_{q'} u = \chi(2^{-q}D)u.
\end{aligned} \tag{3.1}$$

Then we have the formal decomposition

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j\, u, \quad \forall\, u \in \mathcal{S}'_h \quad \text{and} \quad u = \sum_{q\ge -1} \Delta_q\, u \quad \forall\, u \in \mathcal{S}'. \tag{3.2}$$

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_j \dot{\Delta}_k u \equiv 0 \quad \text{if} \quad |j-k| \ge 2 \quad \text{and} \quad \dot{\Delta}_j(\dot{S}_{k-1}u\dot{\Delta}_k u) \equiv 0 \quad \text{if} \quad |j-k| \ge 5. \tag{3.3}$$

We recall now the definition of homogeneous Besov spaces, $\dot{B}^s_{p,r}$, and Bernstein type inequalities from [7]. Similar definition of $B^s_{p,r}$ in the inhomogeneous context can be found in [7].

Definition 3.1 (Definition 2.15 of [7])**.** *Let $(p,r) \in [1,+\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}'_h(\mathbb{R}^2)$, we set*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \Big(2^{js}\|\dot{\Delta}_j u\|_{L^p}\Big)_{\ell^r}.$$

- *For $s < \frac{2}{p}$ (or $s = \frac{2}{p}$ if $r = 1$), we define $\dot{B}^s_{p,r}(\mathbb{R}^2) \stackrel{\text{def}}{=} \big\{u \in \mathcal{S}'_h(\mathbb{R}^2) \;\big|\; \|u\|_{\dot{B}^s_{p,r}} < \infty\big\}$.*
- *If $k \in \mathbb{N}$ and $\frac{2}{p} + k \le s < \frac{2}{p} + k + 1$ (or $s = \frac{2}{p} + k + 1$ if $r = 1$), then $\dot{B}^s_{p,r}(\mathbb{R}^2)$ is defined as the subset of distributions $u \in \mathcal{S}'_h(\mathbb{R}^2)$ such that $\partial^\beta u \in \dot{B}^{s-k}_{p,r}(\mathbb{R}^2)$ whenever $|\beta| = k$.*

In particular, when $p = r = 2$, we obtain the classical Sobolev space, namely, $\dot{H}^s = \dot{B}^s_{2,2}$. Similarly, one has $H^s = B^s_{2,2}$.

Lemma 3.1. *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ an annulus of $\mathbb{R}^2$. A constant C exists so that for any positive real number δ, any non-negative integer k, any smooth homogeneous function σ of degree m, and* *any couple of real numbers $(a,\ b)$ with $b \geq a \geq 1$, there hold*

$$
\begin{aligned}
&\text{Supp } \hat{u} \subset \delta\mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1}\delta^{k+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},\\
&\text{Supp } \hat{u} \subset \delta\mathcal{C} \Rightarrow C^{-1-k}\delta^k\|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k}\delta^k\|u\|_{L^a},\\
&\text{Supp } \hat{u} \subset \delta\mathcal{C} \Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m}\delta^{m+d(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}.
\end{aligned}
\tag{3.4}
$$

We also recall Bony's decomposition from [8]:

$$
uv = T_u v + T'_v u = T_u v + T_v u + R(u,v), \tag{3.5}
$$

where

$$
\begin{aligned}
&T_u v \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j-1}u\dot{\Delta}_j v, \qquad T'_v u \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{S}_{j+2}v\dot{\Delta}_j u,\\
&R(u,v) \overset{\text{def}}{=} \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \dot{\widetilde{\Delta}}_j v \quad \text{with} \quad \dot{\widetilde{\Delta}}_j v \overset{\text{def}}{=} \sum_{|j'-j|\leq 1} \dot{\Delta}_{j'} v.
\end{aligned}
$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin-Lerner type spaces $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})$ from [7].

Definition 3.2. *Let $(r,\lambda,p) \in [1,\ +\infty]^3$ and $T \in]0,\ +\infty]$. We define $\widetilde{L}^\lambda_T(\dot{B}^s_{p\,r}(\mathbb{R}^2))$ as the completion of $C([0,T];\ \mathcal{S}(\mathbb{R}^2))$ by the norm*

$$
\|f\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \overset{def}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{jrs}\Big(\int_0^T \|\dot{\Delta}_j\, f(t)\|^\lambda_{L^p}\, dt\Big)^{\frac{r}{\lambda}}\Big)^{\frac{1}{r}} < \infty.
$$

with the usual change if $r = \infty$. For short, we just denote this space by $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})$.

3.2. **Preliminary estimates.** As an application of the above basic facts on Littlewood-Paley theory, we prove the following Lemmas:

Lemma 3.2. *Let v, w be two smooth divergence free vector fields on $[0,T]\times\mathbb{R}^2$ and ζ, Θ be smooth enough functions on $[0,T]\times\mathbb{R}^2$ which verify*

$$
\begin{cases} \partial_t w + v\cdot\nabla w - \operatorname{div}\big(2\mu(\Theta)d(w)\big) + \nabla\Pi = \zeta e_2 + F,\\ w|_{t=0} = w_0.\end{cases} \tag{3.6}
$$

Then for $p > 4$, one has

$$
\begin{aligned}
\|w\|_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})} \leq & C\Big(\|w_0\|_{\dot{B}^{1/2}_{2,\infty}} + \|\zeta\|_{L^{\frac{4}{3}}_t(L^2)} + \|F\|_{\widetilde{L}^2_t(\dot{B}^{-1/2}_{2,\infty})} + \|\nabla v\|_{L^2_t(L^4)}\|w\|_{L^\infty_t(L^2)}\\
&+ \|\mu(\Theta)-1\|_{L^\infty_t(L^\infty)}\|w\|_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})} + \|\Theta\|^{\frac{p}{p-4}}_{L^\infty_t(\dot{B}^{1/2}_{p,\infty})}\|\nabla w\|_{L^2_t(L^2)}\Big).
\end{aligned}
\tag{3.7}
$$

Proof. We get, by first applying $\dot{\Delta}_j$ to (3.6) and then taking L^2 inner product of the resulting equation with $\dot{\Delta}_j w$, that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j w(t)\|^2_{L^2} + \|\nabla\dot{\Delta}_j w\|^2_{L^2} = &\big([v\cdot\nabla;\dot{\Delta}_j]w \mid \dot{\Delta}_j w\big)\\
&+ 2\big(\operatorname{div}\dot{\Delta}_j\big((\mu(\Theta)-1)d(w)\big) \mid \dot{\Delta}_j w\big) + \big(\dot{\Delta}_j\zeta e_2 \mid \dot{\Delta}_j w\big) + \big(\dot{\Delta}_j F \mid \dot{\Delta}_j w\big),
\end{aligned}
$$

from which and Lemma 3.1, we infer

$$
\begin{aligned}
\|\dot{\Delta}_j w(t)\|_{L^2} \lesssim e^{-c2^{2j}t}\|\dot{\Delta}_j w_0\|_{L^2} + \int_0^t e^{-c2^{2j}(t-t')}\Big(&\|[v\cdot\nabla;\dot{\Delta}_j]w(t')\|_{L^2}\\
&+ 2^j\|\dot{\Delta}_j\big((\mu(\Theta)-1)d(w)\big)(t')\|_{L^2} + \|\dot{\Delta}_j\zeta(t')\|_{L^2} + \|\dot{\Delta}_j F(t')\|_{L^2}\Big)\,dt'.
\end{aligned}
$$

In view of Definition 3.2, by taking L^2 norm of the above inequality over $[0,t]$, then multiplying the resulting inequality by $2^{\frac{3j}{2}}$, and finally taking supreme of j over $\mathbb{Z}$, we obtain

$$
\begin{aligned}
\|w\|_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})} \lesssim \|w_0\|_{\dot{B}^{1/2}_{2,\infty}} &+ \sup_{j\in\mathbb{Z}} 2^{-\frac{j}{2}} \|[\dot{\Delta}_j; v\cdot\nabla]w\|_{L^2_t(L^2)} \\
&+ \|(\mu(\Theta)-1)\nabla u\|_{\widetilde{L}^2_t(\dot{B}^{1/2}_{2,\infty})} + \|\zeta\|_{\widetilde{L}^{4/3}_t(\dot{B}^0_{2,\infty})} + \|F\|_{\widetilde{L}^2_t(\dot{B}^{-1/2}_{2,\infty})}.
\end{aligned} \tag{3.8}
$$

Let us handle term by term above. It is easy to observe that

$$
\|\zeta\|_{\widetilde{L}^{\frac{4}{3}}_t(\dot{B}^0_{2,\infty})} \leq \|\zeta\|_{L^{\frac{4}{3}}_t(L^2)}. \tag{3.9}
$$

Whereas by using Bony's decomposition (3.5), one has

$$
[\dot{\Delta}_j; v\cdot\nabla]w = [\dot{\Delta}_j; T_v\cdot\nabla]w + \dot{\Delta}_j T'_{\nabla w} v - T'_{\dot{\Delta}_j\nabla w} v.
$$

Applying the standard commutator's estimate ([7]) and Lemma 3.1 yields

$$
\begin{aligned}
\|[\dot{\Delta}_j; T_v\cdot\nabla]w(t')\|_{L^2} &\lesssim 2^{-j}\sum_{|j'-j|\leq 4} \|S_{j'-1}\nabla v(t')\|_{L^\infty}\|\dot{\Delta}_{j'}\nabla w(t')\|_{L^2} \\
&\lesssim c_j(t')2^{\frac{j}{2}}\|\nabla v(t')\|_{L^4}\|w(t')\|_{L^2}.
\end{aligned}
$$

While since $\operatorname{div} v = 0$, we have $T'_{\nabla w} v = \operatorname{div}\big(T'_w v\big)$, so that applying Lemma 3.1 gives rise to

$$
\begin{aligned}
\|\dot{\Delta}_j T'_{\nabla w} v(t')\|_{L^2} \lesssim & 2^j \sum_{j'\geq j-N_0} \|\dot{S}_{j'+2} w(t')\|_{L^4}\|\dot{\Delta}_{j'} v(t')\|_{L^4} \\
\lesssim & 2^j \sum_{j'\geq j-N_0} 2^{-\frac{j}{2}} c_{j'}(t')\|w(t')\|_{L^2}\|\nabla v(t')\|_{L^4} \\
\lesssim & c_j(t')2^{\frac{j}{2}}\|\nabla v(t')\|_{L^4}\|w(t')\|_{L^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\|T'_{\dot{\Delta}_j\nabla w} v(t')\|_{L^2} &\lesssim \sum_{j'\geq j} \|\dot{S}_{j'+2}\dot{\Delta}_j\nabla w(t')\|_{L^4}\|\dot{\Delta}_{j'} v(t')\|_{L^4} \\
&\lesssim \|\dot{\Delta}_j\nabla w(t')\|_{L^4}\sum_{j'\geq j}\|\dot{\Delta}_{j'} v(t')\|_{L^4} \lesssim c_j(t')2^{\frac{j}{2}}\|\nabla v(t')\|_{L^4}\|w(t')\|_{L^2}.
\end{aligned}
$$

We thus obtain by using Minkowski inequality that

$$
\begin{aligned}
\sup_{j\in\mathbb{Z}} 2^{-\frac{j}{2}}\|[\dot{\Delta}_j; v\cdot\nabla]w\|_{L^2_t(L^2)} \lesssim & \big\|\big(2^{-\frac{j}{2}}\|[\dot{\Delta}_j; v\cdot\nabla]w(t')\|_{L^2}\big)_{\ell^\infty(Z)}\big\|_{L^2(0,t)} \\
\lesssim & \big\|\big(2^{-\frac{j}{2}}\|[\dot{\Delta}_j; v\cdot\nabla]w(t')\|_{L^2}\big)_{\ell^2(Z)}\big\|_{L^2(0,t)} \\
\lesssim & \|\nabla v\|_{L^2_t(L^4)}\|w\|_{L^\infty_t(L^2)}.
\end{aligned} \tag{3.10}
$$

On the other hand, by using Bony's decomposition (3.5) and para-product estimates ([7]), one has

$$
\|(\mu(\Theta)-1)\nabla w\|_{\widetilde{L}^2_t(\dot{B}^{\frac{1}{2}}_{2,\infty})} \lesssim \|\mu(\Theta)-1\|_{L^\infty_t(L^\infty)}\|w\|_{\widetilde{L}^2_t(\dot{B}^{\frac{3}{2}}_{2,\infty})} + \|\Theta\|_{L^\infty_t(\dot{B}^{1/2}_{p,\infty})}\|\nabla w\|_{L^2_t(L^{\frac{2p}{p-2}})}.
$$

However since $p>4$, one has $\frac{2p}{p-2}>2$ and

$$
\|\nabla w\|_{L^2_t(L^{\frac{2p}{p-2}})} \lesssim \|\nabla w\|_{L^2_t(\dot{B}^0_{\frac{2p}{p-2},2})} \lesssim \|w\|_{L^2_t(\dot{H}^{1+\frac{2}{p}})} \lesssim \|\nabla w\|^{1-\frac{4}{p}}_{L^2_t(L^2)}\|w\|^{\frac{4}{p}}_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})},
$$

so that there holds

$$\|(\mu(\Theta)-1)\nabla w\|_{\widetilde{L}^2_t(\dot{B}^{1/2}_{2,\infty})}\leq C\Big(\|\mu(\Theta)-1\|_{L^\infty_t(L^\infty)}\|w\|_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})}+\|\Theta\|^{\frac{p}{p-4}}_{L^\infty_t(\dot{B}^{1/2}_{p,\infty})}\|\nabla w\|_{L^2_t(L^2)}\Big)+\frac{1}{2}\|w\|_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})}. \tag{3.11}$$

Resuming (3.9), (3.10) and (3.11) into (3.8) leads to (3.7). This completes the proof of Lemma 3.2. □

Lemma 3.3. *Let v be a smooth solenoidal vector field and Θ be a smooth enough function. Then for $s<2$, there holds*

$$\sum_{j\in\mathbb{Z}}2^{-js}\|[\dot{\Delta}_j;v\cdot\nabla]\Theta\|_{L^2}\leq C\|\Theta\|_{\dot{H}^{1-s}}\|\nabla v\|_{L^2}. \tag{3.12}$$

Proof. We first get, by using Bony's decomposition (3.5), that

$$[\dot{\Delta}_j;v\cdot\nabla]\Theta=[\dot{\Delta}_j;T_v]\cdot\nabla\Theta+\dot{\Delta}_jT_{\nabla\Theta}v+\dot{\Delta}_jR(v,\nabla\Theta)-T_{\nabla\dot{\Delta}_j\Theta}v-R(v,\dot{\Delta}_j\nabla\Theta). \tag{3.13}$$

It follows from the classical commutator's estimate (see [7]) and Lemma 3.1 that

$$\begin{aligned}\|[\dot{\Delta}_j;T_v]\nabla\Theta\|_{L^2}&\lesssim 2^{-j}\sum_{|j'-j|\leq 4}\|\dot{S}_{j'-1}\nabla v\|_{L^\infty}\|\dot{\Delta}_{j'}\nabla\Theta\|_{L^2}\\&\lesssim d_j2^{js}\|\Theta\|_{\dot{H}^{1-s}}\|\nabla v\|_{L^2}.\end{aligned}$$

Applying Lemma 3.1 once again gives

$$\begin{aligned}\|T_{\nabla\dot{\Delta}_j\Theta}v\|_{L^2}&\lesssim\sum_{j'\geq j}\|\dot{S}_{j'-1}\dot{\Delta}_j\nabla\Theta\|_{L^\infty}\|\dot{\Delta}_{j'}v\|_{L^2}\\&\lesssim\sum_{j'\geq j}2^{-j'}\|\dot{\Delta}_{j'}\nabla v\|_{L^2}\|\dot{\Delta}_j\nabla\Theta\|_{L^\infty}\lesssim d_j2^{js}\|\Theta\|_{\dot{H}^{1-s}}\|\nabla v\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}\|\dot{\Delta}_jT_{\nabla\Theta}v\|_{L^2}&\lesssim\sum_{|j'-j|\leq 4}\|\dot{S}_{j'-1}\nabla\Theta\|_{L^\infty}\|\dot{\Delta}_{j'}v\|_{L^2}\\&\lesssim d_j2^{js}\|\Theta\|_{\dot{H}^{1-s}}\|\nabla v\|_{L^2}.\end{aligned}$$

The same estimate holds for $R(v,\dot{\Delta}_j\nabla\Theta)$. Finally as $\operatorname{div}v=0$, we write

$$\dot{\Delta}_jR(v,\nabla\Theta)=\sum_{j'\geq j-3}\sum_{k=1}^{2}\dot{\Delta}_j\partial_k(\dot{\Delta}_{j'}v^k\widetilde{\dot{\Delta}}_{j'}\Theta), \tag{3.14}$$

since $s<2$, we get, by applying Lemma 3.1, that

$$\begin{aligned}\|\dot{\Delta}_jR(v,\nabla\Theta)\|_{L^2}&\lesssim 2^{2j}\sum_{j'\geq j-3}\|\dot{\Delta}_{j'}v\|_{L^2}\|\widetilde{\dot{\Delta}}_{j'}\Theta\|_{L^2}\\&\lesssim d_j2^{js}\|\Theta\|_{\dot{H}^{1-s}}\|\nabla v\|_{L^2}.\end{aligned}$$

This completes the proof of (3.12). □

Lemma 3.4. *Let $p>2$ and $\Theta\in\dot{B}^0_{\frac{2p}{p-2},\infty}$. Let $v\in\dot{W}^{1,4}$ be a solenoidal vector filed. Then one has*

$$\sup_{j\in\mathbb{Z}}2^{-\frac{j}{2}}\|[\dot{\Delta}_j;v\cdot\nabla]\Theta\|_{L^{\frac{2p}{p-2}}}\lesssim\|\Theta\|_{\dot{B}^0_{\frac{2p}{p-2},\infty}}\|\nabla v\|_{L^4}. \tag{3.15}$$

Proof. By using Bony's decomposition, we have (3.13). Applying Lemma 3.1 gives

$$\|\dot S_{j'-1}\nabla v\|_{L^\infty}\lesssim\sum_{k\le j'-2}2^{\frac{k}{2}}\|\dot\Delta_k\nabla v\|_{L^4}\lesssim 2^{\frac{j'}{2}}\|\nabla v\|_{L^4},$$

we thus deduce from the classical commutator's estimate (see [7]) that

$$\begin{aligned}\|[\dot\Delta_j;T_v]\nabla\Theta\|_{L^{\frac{2p}{p-2}}}&\lesssim 2^{-j}\sum_{|j'-j|\le 4}\|\dot S_{j'-1}\nabla v\|_{L^\infty}\|\dot\Delta_{j'}\nabla\Theta\|_{L^{\frac{2p}{p-2}}}\\&\lesssim 2^{\frac{j}{2}}\|\nabla v\|_{L^4}\|\Theta\|_{\dot B^0_{\frac{2p}{p-2},\infty}}.\end{aligned}$$

While due to the support property of the Fourier transform to the term $\dot S_{j'-1}\dot\Delta_j\nabla\Theta$, and using Lemma 3.1, one has

$$\begin{aligned}\|T_{\nabla\dot\Delta_j\Theta}v\|_{L^{\frac{2p}{p-2}}}&\lesssim\sum_{j'\ge j}\|\dot\Delta_j\nabla\Theta\|_{L^{\frac{2p}{p-2}}}\|\dot\Delta_{j'}v(t')\|_{L^\infty}\\&\lesssim 2^j\|\Theta\|_{\dot B^0_{\frac{2p}{p-2},\infty}}\sum_{j'\ge j}2^{-\frac{j'}{2}}\|\dot\Delta_{j'}\nabla v\|_{L^4}\lesssim 2^{\frac{j}{2}}\|\Theta\|_{\dot B^0_{\frac{2p}{p-2},\infty}}\|\nabla v\|_{L^4}.\end{aligned}$$

By the same manner, applying Lemma 3.1 once again gives rise to

$$\begin{aligned}\|\dot\Delta_jT_{\nabla\Theta}v\|_{L^{\frac{2p}{p-2}}}&\lesssim\sum_{|j'-j|\le 4}2^{-\frac{j'}{2}}\|\dot\Delta_{j'}\nabla v\|_{L^4}\sum_{k\le j'-2}2^k\|\dot\Delta_k\Theta\|_{L^{\frac{2p}{p-2}}}\\&\lesssim 2^{\frac{j}{2}}\|\Theta\|_{\dot B^0_{\frac{2p}{p-2},\infty}}\|\nabla v\|_{L^4}.\end{aligned}$$

The same estimate holds for $R(v,\dot\Delta_j\nabla\Theta)$.

Finally since $\operatorname{div}v=0$, we have (3.14), from which and Lemma 3.1, we infer

$$\begin{aligned}\|\dot\Delta_jR(v,\nabla\Theta)\|_{L^{\frac{2p}{p-2}}}&\lesssim 2^j\sum_{j'\ge j-3}\|\dot\Delta_{j'}v\|_{L^\infty}\|\widetilde{\dot\Delta}_{j'}\Theta\|_{L^{\frac{2p}{p-2}}}\\&\lesssim 2^j\sum_{j'\ge j-3}2^{-\frac{j'}{2}}\|\dot\Delta_{j'}\nabla v\|_{L^4}\|\widetilde{\dot\Delta}_{j'}\Theta\|_{L^{\frac{2p}{p-2}}}\lesssim 2^{\frac{j}{2}}\|\Theta\|_{\dot B^0_{\frac{2p}{p-2},\infty}}\|\nabla v\|_{L^4}.\end{aligned}$$

Therefore in view of (3.13), we conclude the proof of (3.15). □

As consequence of the previous Lemma, we have following Proposition:

Proposition 3.1. *Let $p>2$ and $\Theta_0\in\dot B^0_{\frac{2p}{p-2},\infty}$, let $v\in L^2_T(\dot W^{1,4})$ be a solenoidal vector field and $f\in\widetilde L^2_t(\dot B^{-1/2}_{\frac{2p}{p-2},\infty})$. Then the equation below*

$$\partial_t\Theta+(v\cdot\nabla)\Theta+|D|\Theta=f\qquad\text{and}\qquad\Theta_{|t=0}=\Theta_0,\tag{3.16}$$

has a unique solution Θ so that for $t\le T$

$$\|\Theta\|_{L^\infty_t(\dot B^0_{\frac{2p}{p-2},\infty})}\le C\Big(\|\Theta_0\|_{\dot B^0_{\frac{2p}{p-2},\infty}}+\|f\|_{L^2_t(\dot B^{-1/2}_{\frac{2p}{p-2},\infty})}\Big)\exp\big(C\|\nabla v\|_{L^2_t(L^4)}\big).\tag{3.17}$$

Proof. As both the existence and uniqueness of solution to (3.16) essentially follows from the *a priori* estimate (3.17). For the sake of simplicity, here we just present the detailed derivation of (3.17) for smooth enough solutions of (3.16). We first get, by applying $\dot\Delta_j$ to the Θ equation

$$\partial_t\dot\Delta_j\Theta+(v\cdot\nabla)\dot\Delta_j\Theta+|D|\dot\Delta_j\Theta=\dot\Delta_jf+[\dot\Delta_j;v\cdot\nabla]\Theta.$$

Taking the L^2 inner product of the previous equation with $|\dot\Delta_j\Theta|^{\frac{4}{p-2}}\dot\Delta_j\Theta$ and using the generalized Bernstein inequality in [20, 31], we obtain

$$
\begin{aligned}
&\frac{p-2}{2p}\frac{d}{dt}\|\dot\Delta_j\Theta(t)\|_{L^{\frac{2p}{p-2}}}^{\frac{2p}{p-2}}+c2^j\|\dot\Delta_j\Theta(t)\|_{L^{\frac{2p}{p-2}}}^{\frac{2p}{p-2}}\\
&\qquad\qquad\leq C\|\dot\Delta_j\theta(t)\|_{L^{\frac{2p}{p-2}}}^{\frac{p+2}{p-2}}\big(\|\dot\Delta_j f(t)\|_{L^{\frac{2p}{p-2}}}+\|[\dot\Delta_j;v\cdot\nabla]\Theta\|_{L^{\frac{2p}{p-2}}}\big).
\end{aligned}\tag{3.18}
$$

Let $J=:[J^-,J^+]$ be a subinterval of $[0,T]$. Then for $t\in J$, we deduce from (3.18) that

$$
\begin{aligned}
\|\dot\Delta_j\Theta(t)\|_{L^{\frac{2p}{p-2}}}\leq C\Big(&e^{-ct2^j}\|\dot\Delta_j\Theta(J^-)\|_{L^{\frac{2p}{p-2}}}\\
&+\int_{J^-}^t e^{-c(t-t')2^j}\big(\|\dot\Delta_j f\|_{L^{\frac{2p}{p-2}}}+\|[\dot\Delta_j;v\cdot\nabla]\Theta\|_{L^{\frac{2p}{p-2}}}\big)(t')\,dt'\Big),
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\Theta\|_{\widetilde L^\infty(J;\dot B^0_{\frac{2p}{p-2},\infty})}\leq C\Big(&\|\Theta(J^-)\|_{\dot B^0_{\frac{2p}{p-2},\infty}}+\|f\|_{\widetilde L^2(J;\dot B^{-1/2}_{\frac{2p}{p-2},\infty})}\\
&+\sup_{j\in\mathbb Z}2^{-\frac j2}\|[\dot\Delta_j;v\cdot\nabla]\Theta\|_{L^2(J;L^{\frac{2p}{p-2}})}\Big).
\end{aligned}
$$

However applying Lemma 3.4 and Minkowski inequality gives

$$
\begin{aligned}
\sup_{j\in\mathbb Z}2^{-\frac j2}\|[\dot\Delta_j;v\cdot\nabla]\Theta\|_{L^2(J;L^{\frac{2p}{p-2}})}&\lesssim\Big\|\sup_{j\in\mathbb Z}2^{-\frac j2}\|[\dot\Delta_j;v\cdot\nabla]\Theta\|_{L^{\frac{2p}{p-2}}}\Big\|_{L^2(J)}\\
&\lesssim\|\nabla v\|_{L^2(J;L^4)}\|\Theta\|_{L^\infty(J;\dot B^0_{\frac{2p}{p-2},\infty})}.
\end{aligned}
$$

We thus obtain

$$
\|\Theta\|_{\widetilde L^\infty(J;\dot B^0_{\frac{2p}{p-2},\infty})}\leq C\Big(\|\Theta(J^-)\|_{\dot B^0_{\frac{2p}{p-2},\infty}}+\|f\|_{L^2(J;\dot B^{-1/2}_{\frac{2p}{p-2},\infty})}+\|\nabla v\|_{L^2(J;L^4)}\|\Theta\|_{L^\infty(J;\dot B^0_{\frac{2p}{p-2},\infty})}\Big).
$$

Let us take $|J|=J^+-J^-$ to be so small that

$$
C\|\nabla v\|_{L^2(J;L^4)}\leq 1/2,
$$

then we obtain

$$
\|\Theta\|_{\widetilde L^\infty(J;\dot B^0_{\frac{2p}{p-2},\infty})}\leq 2C\Big(\|\Theta(J^-)\|_{\dot B^0_{\frac{2p}{p-2},\infty}}+\|f\|_{L^2(J;\dot B^{-1/2}_{\frac{2p}{p-2},\infty})}\Big).\tag{3.19}
$$

Now let us decompose $[0,t]$ into subintervals, $J_k=\big[t_k,t_{k+1}\big[$, $k=0,\cdots,M-1$ so that $[0,t]=\cup_{k=0}^{M-1}J_k$ with $t_0=0,t_M=t$ and

$$
\forall k\in\big\{0,1,\cdots,M-2\big\},\quad C\|\nabla v\|_{L^2(J_k;L^4)}=1/2\quad\text{and}\quad C\|\nabla v\|_{L^2(J_{M-1};L^4)}\leq 1/2.
$$

Let us observe that

$$
M\leq 2+2C\|\nabla v\|_{L^2_t(L^4)}.\tag{3.20}
$$

On the other hand, it follows from (3.19) that

$$
\begin{aligned}
\|\Theta\|_{\widetilde L^\infty(J_k;\dot B^0_{\frac{2p}{p-2},\infty})}&\leq 2C\Big(\|\Theta(t_k)\|_{\dot B^0_{\frac{2p}{p-2},\infty}}+\|f\|_{L^2(J_k;\dot B^{-1/2}_{\frac{2p}{p-2},\infty})}\Big)\\
&\leq 2C\Big(\|\Theta\|_{\widetilde L^\infty(J_{k-1};\dot B^0_{\frac{2p}{p-2},\infty})}+\|f\|_{L^2(J_k;\dot B^{-1/2}_{\frac{2p}{p-2},\infty})}\Big).
\end{aligned}
$$

Inductively this gives rise to

$$\begin{aligned}\|\Theta\|_{\widetilde{L}^\infty(J_k;\dot{B}^0_{\frac{2p}{p-2},\infty})} &\leq (2C)^k\Big(\|\Theta\|_{\widetilde{L}^\infty(J_0;\dot{B}^0_{\frac{2p}{p-2},\infty})}+\sum_{\ell=0}^k\|f\|_{L^2(J_\ell;\dot{B}^{-1/2}_{\frac{2p}{p-2},\infty})}\Big)\\ &\leq (2C)^k\Big(\|\Theta_0\|_{\dot{B}^0_{\frac{2p}{p-2},\infty}}+\sqrt{k}\|f\|_{L^2(\cup_{\ell=0}^k J_\ell;\dot{B}^{-1/2}_{\frac{2p}{p-2},\infty})}\Big),\end{aligned}$$

which together with (3.20) ensures (3.17). This completes the proof of the proposition. □

4. The key *a priori* Estimates

4.1. **The basic energy estimate for θ and u.** In this subsection, we shall present the time decay estimate of $\|\theta(t)\|_{L^2}$ and the energy estimate for u.

Lemma 4.1. *Let (θ,u) be a smooth enough solution of the System (1.1) on $[0,T^*[$. Let $h(\cdot)\in C^1$ with $h^2(\cdot)$ being convex function of its variable. Then we have*

$$\|h(\theta(t))\|_{L^p}\leq\|h(\theta_0)\|_{L^p}\quad\forall\,p\in[2,\infty]\quad\text{and}\quad t<T^*. \tag{4.1}$$

Proof. Since $h^2(\cdot)$ is a convex function, we deduce from (1.1) of [13] that

$$\frac{1}{2}|D|(h^2(\theta))(t,x)\leq h(\theta(t,x))h'(\theta(t,x))|D|\theta(t,x),$$

from which, we infer

$$\partial_t h^2(\theta)+u\cdot\nabla h^2(\theta)+|D|h^2(\theta)\leq 2h(\theta)h'(\theta)\big(\partial_t\theta+u\cdot\nabla\theta+|D|\theta\big)=0. \tag{4.2}$$

While we get, by using a similar proof of Lemma 2.5 of [12], that

$$\begin{aligned}&\int_{\mathbb{R}^2}|h(\theta)|^{p-2}|D|h^2(\theta)\,dx\\ &=\frac{1}{2\pi}\lim_{\varepsilon\to0}\int_{\mathbb{R}^2}\int_{|x-y|\geq\varepsilon}|h(\theta(t,x))|^{p-2}\frac{h^2(\theta(t,x))-h^2(\theta(t,y))}{|x-y|^3}\,dy\,dx\\ &=-\frac{1}{2\pi}\lim_{\varepsilon\to0}\int_{\mathbb{R}^2}\int_{|x-y|\geq\varepsilon}|h(\theta(t,y))|^{p-2}\frac{h^2(\theta(t,x))-h^2(\theta(t,y))}{|x-y|^3}\,dy\,dx\\ &=\frac{1}{4\pi}\lim_{\varepsilon\to0}\int_{\mathbb{R}^2}\int_{|x-y|\geq\varepsilon}\big(|h(\theta(t,x))|^{p-2}-|h(\theta(t,y))|^{p-2}\big)\,\frac{h^2(\theta(t,x))-h^2(\theta(t,y))}{|x-y|^3}\,dy\,dx\geq0,\end{aligned}$$

for any $p\in[2,\infty[$. Then for any $p\in[2,\infty[$, multiplying (4.2) by $\frac{p}{2}|h(\theta)|^{p-2}$ and integrating the resulting inequality over $\mathbb{R}^2$, we obtain

$$\frac{d}{dt}\|h(\theta(t,\cdot))\|_{L^p}^p+\int_{\mathbb{R}^2}u\cdot\nabla|h(\theta)|^p\,dx\leq0,$$

which together with $\operatorname{div}u=0$ ensures

$$\|h(\theta(t,\cdot))\|_{L^p}\leq\|h(\theta_0)\|_{L^p}\quad\forall\;p\in[2,\infty[,$$

which implies also (4.1) for $p=\infty$. This completes the proof of the lemma. □

Proposition 4.1. *Let $q\in]1,2[$ and $s_0\in\big[2/q-1,2(2/q-1)\big[$. Let (θ,u) be a smooth enough solution of the System (1.1) on $[0,T^*[$. We assume that $(\theta_0,u_0)\in\big(L^q\cap L^2\cap\dot{H}^{-s_0}\big)\times L^2$, then (1.7) holds for any $t<T^*$. If moreover, $q\in]1,4/3[$ and $s_0\in\big]1,2(2/q-1)\big[$, we have (2.1) for any $t<T^*$.*

Proof. We first get, by taking L^2 inner product of u equation of (1.1) with u, that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2+\int_{\mathbb{R}^2}\mu(\theta)d:d\,dx=\int_{\mathbb{R}^2}\theta u_2\,dx, \tag{4.3}$$

from which and (1.4), we deduce that for $1<q<2$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2 &\leq \|\theta\|_{L^q}\|u\|_{L^{\frac{q}{q-1}}}\leq C\|\theta\|_{L^q}\|u\|_{L^2}^{\frac{2(q-1)}{q}}\|\nabla u\|_{L^2}^{\frac{2-q}{q}}\\ &\leq C\|\theta\|_{L^q}^{\frac{2q}{3q-2}}\|u\|_{L^2}^{\frac{4(q-1)}{3q-2}}+\frac{1}{2}\|\nabla u\|_{L^2_t(L^2)}^2.\end{aligned}$$

Applying Osgood's Lemma gives

$$\Bigl(\|u(t)\|_{L^2}^2+\|\nabla u\|_{L^2_t(L^2)}^2\Bigr)^{\frac{q}{3q-2}}\leq\|u_0\|_{L^2}^{\frac{2q}{3q-2}}+\frac{q}{3q-2}\int_0^t\|\theta(t')\|_{L^q}^{\frac{2q}{3q-2}}\,dt',$$

which implies

$$\|u\|_{L^\infty_t(L^2)}^2+\|\nabla u\|_{L^2_t(L^2)}^2\leq\|u_0\|_{L^2}^2+C\|\theta\|^2_{L_t^{\frac{2q}{3q-2}}(L^q)}.$$

Then by virtue of Lemma 4.1, we thus obtain

$$\|u\|_{L^\infty_t(L^2)}+\|\nabla u\|_{L^2_t(L^2)}\leq C\bigl(\|u_0\|_{L^2}+\|\theta_0\|_{L^q}t^{\frac{3q-2}{2q}}\bigr). \tag{4.4}$$

On the other hand, we get, by taking L^2 inner product of the temperature equation in (1.1) with θ, that

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{L^2}^2+\|\theta(t)\|_{\dot H^{\frac12}}^2=0. \tag{4.5}$$

Motivated by Schonbek's strategy for the classical Navier-Stokes system in [28] (see also [30]), we split the phase-space $\mathbb{R}^2$ into two time-dependent regions $S(t)\stackrel{\text{def}}{=}\bigl\{\xi\in\mathbb{R}^2,\ |\xi|\leq g(t)\bigr\}$ and $S^c(t)$, the complement of the set $S(t)$ in $\mathbb{R}^2$, for some $g(t)\sim\langle t\rangle^{-1}$ to be determined hereafter. Then we deduce from (4.5) that

$$\frac{d}{dt}\|\theta(t)\|_{L^2}^2+2g(t)\|\theta(t)\|_{L^2}^2\leq 2g(t)\int_{S(t)}|\widehat{\theta}(t,\xi)|^2\,d\xi. \tag{4.6}$$

In order to deal with the source term in (4.6), we rewrite the θ equation of (1.1) as

$$\widehat{\theta}(t,\xi)=e^{-t|\xi|}\widehat{\theta}_0(\xi)-\int_0^t e^{-(t-t')|\xi|}\xi\cdot\mathcal{F}_x(\theta u)(t',\xi)\,dt'.$$

Since $\theta_0\in\dot H^{-s_0}(\mathbb{R}^2)$, one has

$$\int_{S(t)}\bigl|e^{-t|\xi|}\widehat{\theta}_0(\xi)\bigr|^2\,d\xi\leq\langle t\rangle^{-2s_0}\|\theta_0\|_{\dot H^{-s_0}}^2.$$

While it follows from Young's inequality that

$$\begin{aligned}\int_{S(t)}\Bigl|\int_0^t e^{-(t-t')|\xi|}\xi\cdot\mathcal{F}_x(\theta u)(t',\xi)\,dt'\Bigr|^2\,d\xi\leq& Cg^4(t)\Bigl\|\int_0^t e^{-(t-t')|\xi|}\mathcal{F}_x(\theta u)(t',\xi)\,dt'\Bigr\|_{L^\infty}^2\\ \leq& Cg^4(t)\Bigl(\int_0^t\|(\theta u)(t',\cdot)\|_{L^1}\,dt'\Bigr)^2\\ \leq& Cg^4(t)\|\theta\|_{L^\infty_t(L^q)}^2\|u\|^2_{L^1_t(L^{\frac{q}{q-1}})}.\end{aligned}$$

However note from (4.4) that

$$\begin{aligned}\|u\|_{L^1_t(L^{\frac{q}{q-1}})} \leq & C\langle t\rangle^{\frac32-\frac1q}\|u\|_{L_t^{\frac{2q}{2-q}}(L^{\frac{q}{q-1}})}\\ \leq & C\langle t\rangle^{\frac32-\frac1q}\|u\|_{L^\infty_t(L^2)}^{\frac{2(q-1)}{q}}\|\nabla u\|_{L^2_t(L^2)}^{\frac{2-q}{q}} \leq C\big(\|u_0\|_{L^2}+\|\theta_0\|_{L^q}\big)\langle t\rangle^{3-\frac2q}.\end{aligned}$$

This together with the fact: $g(t)\sim\langle t\rangle^{-1}$, ensures

$$\int_{S(t)}|\widehat{\theta}(t,\xi)|^2\,d\xi \leq C\Big(\langle t\rangle^{-2s_0}\|\theta_0\|^2_{H^{-s_0}}+\|\theta_0\|^2_{L^q}\big(\|u_0\|_{L^2}+\|\theta_0\|_{L^q}\big)^2\langle t\rangle^{2-\frac4q}\Big). \tag{4.7}$$

Resuming the above estimate into (4.6) and using the assumption that $s_0\geq \frac2q-1$, we obtain

$$\frac{d}{dt}\|\theta\|^2_{L^2}+2g(t)\|\theta\|^2_{L^2}\leq C\mathcal{E}_0^2\langle t\rangle^{1-\frac4q}$$

for $\mathcal{E}_0$ given by (1.8). Hence there holds

$$\frac{d}{dt}\Big(\|\theta(t)\|^2_{L^2}\exp\Big(2\int_0^t g(t')\,dt'\Big)\Big)\leq C\mathcal{E}_0^2\langle t\rangle^{1-\frac4q}\exp\Big(2\int_0^t g(t')\,dt'\Big).$$

Let us choose $g(t)=\alpha\langle t\rangle^{-1}$ for $\alpha>2/q-1$ in the above inequality to get

$$\langle t\rangle^{2\alpha}\|\theta(t)\|^2_{L^2}\leq\|\theta_0\|^2_{L^2}+C\mathcal{E}_0^2\langle t\rangle^{2\alpha+2-\frac4q},$$

which implies

$$\|\theta(t)\|_{L^2}\leq C\mathcal{E}_0\langle t\rangle^{1-\frac2q}\quad\text{for}\quad t\in]0,T^*[. \tag{4.8}$$

In view of (4.8) and (4.3), we write

$$\|u(t)\|_{L^2}\leq\|u_0\|_{L^2}+\|\theta\|_{L^1_t(L^2)}\leq\|u_0\|_{L^2}+C\mathcal{E}_0\langle t\rangle^{2-\frac2q}\leq C\mathcal{E}_0\langle t\rangle^{2-\frac2q},$$

so that thanks to (4.8), we get, by a similar derivation of (4.7), that

$$\begin{aligned}\int_{S(t)}|\widehat{\theta}(t,\xi)|^2d\xi &\leq C\Big(\mathcal{E}_0^2\langle t\rangle^{-2s_0}+g(t)^4\Big(\int_0^t\|\theta(t')\|_{L^2}\|u(t')\|_{L^2}\,dt'\Big)^2\Big)\\ &\leq C\Big(\mathcal{E}_0^2\langle t\rangle^{-2s_0}+\mathcal{E}_0^4 g(t)^4\Big(\int_0^t\langle t'\rangle^{3-\frac4q}\,dt'\Big)^2\Big)\\ &\leq C\Big(\mathcal{E}_0^2\langle t\rangle^{-2s_0}+\mathcal{E}_0^4\langle t\rangle^{4-\frac8q}\Big).\end{aligned}$$

Substituting the above inequality into (4.6) and using the assumption that $s_0\leq 2\big(2/q-1\big)$, we arrive at

$$\frac{d}{dt}\|\theta\|^2_{L^2}+2g(t)\|\theta\|^2_{L^2}\leq C\mathcal{E}_0^2\big(1+\mathcal{E}_0^2\big)\langle t\rangle^{-1-2s_0}=CE_0^2\langle t\rangle^{-1-2s_0}.$$

Thus taking $g(t)=\alpha\langle t\rangle^{-1}$ for $\alpha>s_0$ in the above inequality, we get, by using a similar derivation of (4.8), that

$$\langle t\rangle^{2\alpha}\|\theta(t)\|^2_{L^2}\leq C\Big(\|\theta_0\|^2_{L^2}+E_0^2\langle t\rangle^{2\alpha-2s_0}\Big).$$

Divided the above inequality by $\langle t\rangle^{2\alpha}$ leads to (1.7) for $t<T^*$.

By virtue of (1.7), if $s_0\in\big]1,2\big(2/q-1\big)\big[$ for some $q\in\big]1,4/3\big[$, we get

$$\|\theta\|_{L^1_t(L^2)}\leq CE_0, \tag{4.9}$$

from which and (4.3), we infer

$$\|u\|_{L^\infty_t(L^2)}+\|\nabla u\|_{L^2_t(L^2)}\leq\|u_0\|_{L^2}+C\|\theta\|_{L^1_t(L^2)}\leq CE_0,$$

which ensures (2.1). This completes the proof of Proposition 4.1. □

4.2. **The derivative energy estimate for (θ, u).** The purpose of this subsection is to prove the $\dot{H}^{1/2}$ energy estimate for θ and $\dot{H}^1$ energy estimate for u.

Lemma 4.2. *Let (θ, u) be a smooth enough solution of (1.1) on $[0, T^*[$. Then under the assumptions of Proposition 4.1, for any $p > 4$ and any $t < T^*$, we have*

$$\|\theta\|_{L^\infty_t(B^{1/2}_{p,\infty})} + \|\theta\|_{\widetilde{L}^2_t(B^1_{p,\infty})} \leq \|\theta_0\|_{B^{1/2}_{p,\infty}} + C\|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2_t(L^p)}, \tag{4.10}$$

and

$$\begin{aligned}\|\theta\|^2_{\widetilde{L}^\infty_t(\dot{H}^{1/2})} + \|\theta\|^2_{L^2_t(\dot{H}^1)} \leq & C_p\Big(\|\theta_0\|^2_{\dot{H}^{1/2}} + E_0^2\big(1 + \|\theta_0\|^2_{L^2\cap L^\infty}\big) \\ & + E_0^2\|\theta_0\|^2_{L^\infty}\ln\big(e + \|\theta_0\|_{B^{1/2}_{p,\infty}} + \|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2_t(L^p)}\big)\Big)\end{aligned} \tag{4.11}$$

where E_0 is given by (1.8)

Proof. We first get, by applying $\dot{\Delta}_j$ to the θ equation of (1.1) and then taking the L^2 inner product of the resulting equation with $\dot{\Delta}_j\theta$, that

$$\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j\theta(t)\|^2_{L^2} + \big(|D|\dot{\Delta}_j\theta(t) \mid \dot{\Delta}_j\theta(t)\big)_{L^2} = -\big(\dot{\Delta}_j(u\cdot\nabla\theta) \mid \dot{\Delta}_j\theta\big)_{L^2}. \tag{4.12}$$

To estimate the term on the right hand-side of (4.12), we use Bony's decomposition (3.5) to write

$$u\cdot\nabla\theta = T_u\nabla\theta + T_{\nabla\theta}u + R(u, \nabla\theta).$$

Applying Lemma 3.1 gives

$$\begin{aligned}\|\dot{\Delta}_j(T_{\nabla\theta}u)(t)\|_{L^2} &\lesssim \sum_{|j-\ell|\leq 4}\|\dot{S}_{\ell-1}\nabla\theta(t)\|_{L^\infty}\|\dot{\Delta}_\ell u(t)\|_{L^2} \\ &\lesssim \sum_{|j-\ell|\leq 4}\|\dot{S}_{\ell-1}\theta(t)\|_{L^\infty}\|\dot{\Delta}_\ell\nabla u(t)\|_{L^2} \\ &\lesssim c_j(t)\|\theta(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2},\end{aligned}$$

and due to $\operatorname{div} u = 0$, one has

$$\begin{aligned}\|\dot{\Delta}_j(R(u,\nabla\theta)(t)\|_{L^2} \lesssim 2^j\|\dot{\Delta}_j(R(u,\theta))(t)\|_{L^2} &\lesssim 2^j\sum_{\ell\geq j-3}\|\dot{\Delta}_\ell u(t)\|_{L^2}\|\dot{\widetilde{\Delta}}_\ell\theta(t)\|_{L^\infty} \\ &\lesssim 2^j\sum_{\ell\geq j-3}c_\ell(t)2^{-\ell}\|\theta(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2} \\ &\lesssim c_j(t)\|\theta(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2}.\end{aligned}$$

Whereas by using a standard commutator's argument and $\operatorname{div} u = 0$, we write

$$\begin{aligned}\big(\dot{\Delta}_j(T_u\nabla\theta) \mid \dot{\Delta}_j\theta\big)_{L^2} = \sum_{|j-\ell|\leq 4}\Big(&\big([\dot{\Delta}_j; \dot{S}_{\ell-1}u]\nabla\dot{\Delta}_\ell\theta \mid \dot{\Delta}_j\theta\big)_{L^2} \\ &+ \big((\dot{S}_{\ell-1}u - \dot{S}_{j-1}u)\cdot\nabla\dot{\Delta}_\ell\dot{\Delta}_j\theta \mid \dot{\Delta}_j\theta\big)_{L^2}\Big).\end{aligned}$$

It follows from the commutator's estimate ([7]) that

$$\begin{aligned}\sum_{|j-\ell|\leq 4}\big|\big([\dot{\Delta}_j; \dot{S}_{\ell-1}u]\nabla\dot{\Delta}_\ell\theta \mid \dot{\Delta}_j\theta\big)_{L^2}\big| &\lesssim 2^{-j}\sum_{|j-\ell|\leq 4}\|\dot{S}_{\ell-1}\nabla u(t)\|_{L^2}\|\dot{\Delta}_\ell\nabla\theta(t)\|_{L^\infty}\|\dot{\Delta}_j\theta(t)\|_{L^2} \\ &\lesssim c_j^2(t)2^{-j}\|\nabla u(t)\|_{L^2}\|\theta(t)\|_{\dot{B}^0_{\infty,2}}\|\nabla\theta(t)\|_{L^2}.\end{aligned}$$

While applying Lemma 3.1 leads to

$$\sum_{|j-\ell|\leq 4}\big|\big((\dot{S}_{\ell-1}u - \dot{S}_{j-1}u)\cdot\nabla\dot{\Delta}_\ell\dot{\Delta}_j\theta) \mid \dot{\Delta}_j\theta\big)_{L^2}\big| \lesssim c_j^2(t)2^{-j}\|\nabla u(t)\|_{L^2}\|\theta(t)\|_{L^\infty}\|\nabla\theta(t)\|_{L^2}.$$

As a consequence, we obtain

$$\big|\big(\dot\Delta_j(u\cdot\nabla\theta)\mid\dot\Delta_j\theta\big)_{L^2}\big|\lesssim c_j^2(t)2^{-j}\|\nabla u(t)\|_{L^2}\big(\|\theta(t)\|_{L^\infty}+\|\theta(t)\|_{\dot B^0_{\infty,2}}\big)\|\nabla\theta(t)\|_{L^2}.$$

Substituting the above estimate into (4.12), and integrating the resulting inequality over $[0,t]$, we infer

$$\|\theta\|^2_{\widetilde L^\infty_t(\dot H^{1/2})}+\|\theta\|^2_{L^2_t(\dot H^1)}\leq\|\theta_0\|^2_{\dot H^{1/2}}+C\big(\|\theta\|^2_{L^\infty_t(L^\infty)}+\|\theta\|^2_{L^\infty_t(\dot B^0_{\infty,2})}\big)\|\nabla u\|^2_{L^2_t(L^2)}. \tag{4.13}$$

On the other hand, for any $p\in[1,\infty[$, we deduce from the proof of Theorem 4.2 of [23] that for any $q\geq-1$,

$$\|\Delta_q\theta(t)\|_{L^p}\leq\|\Delta_q\theta_0\|_{L^p}\exp\big(-ct2^q\big)+C\|\theta_0\|_{L^\infty}\int_0^t\exp\big(-c(t-t')2^q\big)\|\nabla u(t')\|_{L^p}\,dt',$$

which implies

$$\begin{aligned}\|\Delta_q\theta\|_{L^\infty_t(L^p)}+2^{\frac q2}\|\Delta_q\theta\|_{L^2_t(L^p)}\leq&C\big(\|\Delta_q\theta_0\|_{L^p}+2^{-\frac q2}\|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2_t(L^p)}\big)\\ \leq&C2^{-\frac q2}\big(\|\theta_0\|_{B^{1/2}_{p,\infty}}+\|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2_t(L^p)}\big).\end{aligned}$$

This gives rise to (4.10).

Note that for any positive integer N and $p>4$, we write

$$\begin{aligned}\|\theta\|_{L^\infty_t(\dot B^0_{\infty,2})}\leq&\|\theta\|_{L^\infty_t(L^2)}+\Big(\sum_{1\leq q\leq N}\|\Delta_q\theta\|^2_{L^\infty_t(L^\infty)}\Big)^{\frac12}+\Big(\sum_{q>N}\|\Delta_q\theta\|^2_{L^\infty_t(L^\infty)}\Big)^{\frac12}\\ \leq&\|\theta_0\|_{L^2}+\|\theta_0\|_{L^\infty}\sqrt N+2^{-N\left(\frac12-\frac2p\right)}\|\theta\|_{\widetilde L^\infty_t(B^{1/2}_{p,\infty})}.\end{aligned}$$

Taking N in the above inequality so that

$$2^{N\left(\frac12-\frac2p\right)}\sim\|\theta\|_{\widetilde L^\infty_t(B^{1/2}_{p,\infty})},$$

we obtain for any $p>4$

$$\|\theta\|_{L^\infty_t(\dot B^0_{\infty,2})}\leq C_p\Big(1+\|\theta_0\|_{L^2}+\|\theta_0\|_{L^\infty}\ln^{\frac12}\big(e+\|\theta\|_{\widetilde L^\infty_t(B^{1/2}_{p,\infty})}\big)\Big),$$

which together with (4.10) implies that

$$\|\theta\|_{L^\infty_t(\dot B^0_{\infty,2})}\leq C_p\Big(1+\|\theta_0\|_{L^2}+\|\theta_0\|_{L^\infty}\ln^{\frac12}\big(e+\|\theta_0\|_{B^{1/2}_{p,\infty}}+\|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2_t(L^p)}\big)\Big). \tag{4.14}$$

Resuming the above inequality into (4.13) leads to

$$\begin{aligned}\|\theta\|^2_{\widetilde L^\infty_t(\dot H^{1/2})}+\|\theta\|^2_{L^2_t(\dot H^1)}\leq\|\theta_0\|^2_{\dot H^{1/2}}+C_p\|\nabla u\|^2_{L^2_t(L^2)}\Big(1+\|\theta_0\|^2_{L^2\cap L^\infty}&\\ +\|\theta_0\|^2_{L^\infty}\ln\big(e+\|\theta_0\|_{B^{1/2}_{p,\infty}}+\|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2_t(L^p)}\big)\Big),&\end{aligned}$$

from which and (2.1), we infer (4.11). This completes the proof of Lemma 4.2. $\square$

Lemma 4.3. *Let (θ,u) be a smooth enough solution of (1.1) on $[0,T^*[$. Then under the assumptions of Proposition 4.1 and (1.5), one has for all $t\in[0,T^*[$ and $p\in]4,p^*]$*

$$\|\nabla u\|^2_{L^\infty_t(L^2)}+\|\partial_t u\|^2_{L^2_t(L^2)}\leq C\mathfrak{C}_0\big(1+\|\theta_0\|_{B^{1/2}_{p,\infty}}+\|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2_t(L^p)}\big)^{\delta_1}, \tag{4.15}$$

where p^* is given by (1.4), $\mathfrak{C}_0$ and δ_1 are given by

$$\begin{aligned}&\mathfrak{C}_0\stackrel{\text{def}}{=}\big(E_0^2+\|\nabla u_0\|^2_{L^2}\big)\exp\Big(C\Big(\|\theta_0\|^2_{\dot H^{1/2}}+E_0^2\big(1+\|\theta_0\|^2_{L^2\cap L^\infty}+E_0^2\big)\Big)\Big)\quad\text{and}\\ &\delta_1\stackrel{\text{def}}{=}C\|\theta_0\|^2_{L^\infty}E_0^2\quad\text{for}\quad E_0\quad\text{given by (1.8).}\end{aligned} \tag{4.16}$$

Proof. The proof of this lemma is motivated by that of Theorem 1 in [17] and that of Proposition 2.1 of [5]. In fact, by taking the L^2 inner product of the momentum equation of (1.1) with $\partial_t u$, we write

$$\int_{\mathbb{R}^2} |\partial_t u|^2\,dx - \int_{\mathbb{R}^2} \operatorname{div}\big(2\mu(\theta)d\big) \mid \partial_t u\,dx = -\int_{\mathbb{R}^2} \partial_t u \mid \big(u\cdot\nabla u\big)\,dx + \int_{\mathbb{R}^2} \theta\partial_t u_2\,dx. \tag{4.17}$$

Motivated by the derivation of (29) in [17], we get, by using integration by parts, that

$$\begin{aligned} -\int_{\mathbb{R}^2} \operatorname{div}\big(2\mu(\theta)d\big) \mid \partial_t u\,dx &= \int_{\mathbb{R}^2} 2\mu(\theta)d : \partial_t d\,dx \\ &= \frac{d}{dt}\int_{\mathbb{R}^2} \mu(\theta)d : d\,dx - \int_{\mathbb{R}^2} \partial_t\big(\mu(\theta)\big)d : d\,dx. \end{aligned}$$

Using the θ equation of (1.1) and then integration by parts, we get

$$\begin{aligned} -\int_{\mathbb{R}^2} \partial_t\big(\mu(\theta)\big)d : d\,dx &= \int_{\mathbb{R}^2} \big(u\cdot\nabla\mu(\theta) + \mu'(\theta)|D|\theta\big)d : d\,dx \\ &= -\int_{\mathbb{R}^2} u\cdot\nabla\big(\mu(\theta)\big)^{-1}(\mu(\theta)d) : (\mu(\theta)d)\,dx + \int_{\mathbb{R}^2} \mu'(\theta)\big(|D|\theta\big)d : d\,dx \\ &= \sum_{i=1}^2 \int_{\mathbb{R}^2} u_i d : \partial_i\big(2\mu(\theta)d\big)\,dx + \int_{\mathbb{R}^2} \mu'(\theta)\big(|D|\theta\big)d : d\,dx. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{i=1}^2 \int_{\mathbb{R}^2} u_i d : \partial_i\big(2\mu(\theta)d\big)\,dx &= \sum_{1\le i,k,\ell\le 2} \int_{\mathbb{R}^2} u_i\partial_k u_\ell \partial_i\big(2\mu(\theta)d_{k\ell}\big)\,dx \\ &= -\sum_{1\le i,k,\ell\le 2} \Big(\int_{\mathbb{R}^2} \partial_k u_i u_\ell \partial_i\big(2\mu(\theta)d_{k\ell}\big)\,dx + \int_{\mathbb{R}^2} u_i u_\ell \partial_i\partial_k\big(2\mu(\theta)d_{k\ell}\big)\,dx\Big). \end{aligned}$$

Hence due to $\operatorname{div} u = 0$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\partial_t u|^2\,dx + \frac{d}{dt}\int_{\mathbb{R}^2} \mu(\theta)d : d\,dx &= -\int_{\mathbb{R}^2} \partial_t u \mid \big(u\cdot\nabla u\big)\,dx \\ &\quad + \int_{\mathbb{R}^2} \theta\partial_t u_2\,dx - \sum_{1\le i,k,\ell\le 2} 2\int_{\mathbb{R}^2} \mu(\theta)\partial_k u_i \partial_i u_\ell d_{k,\ell}\,dx \\ &\quad - \sum_{1\le i,k,\ell\le 2} \int_{\mathbb{R}^2} u_i\,\partial_i u_\ell\,\partial_k\big(2\mu(\theta)d_{k\ell}\big)dx - \int_{\mathbb{R}^2} \mu'(\theta)\big(|D|\theta\big)d : d\,dx, \end{aligned}$$

which together with the momentum equation of (1.1) implies that

$$\begin{aligned} &\int_{\mathbb{R}^2} |\partial_t u|^2\,dx + \frac{d}{dt}\int_{\mathbb{R}^2} \mu(\theta)d : d\,dx \\ &\le \|\partial_t u\|_{L^2}\|u\cdot\nabla u\|_{L^2} + \|\theta\|_{L^2}\|\partial_t u_2\|_{L^2} - \int_{\mathbb{R}^2} \mu'(\theta)\big(|D|\theta\big)d : d\,dx \\ &\quad - \int_{\mathbb{R}^2} u\cdot\nabla u \mid \big(\partial_t u + u\cdot\nabla u + \nabla\Pi - \theta e_2\big)\,dx - \sum_{1\le i,k,\ell\le 2} 2\int_{\mathbb{R}^2} \mu(\theta)\partial_k u_i \partial_i u_\ell d_{k,\ell}\,dx. \end{aligned}$$

This gives

$$\begin{aligned} \int_{\mathbb{R}^2} |\partial_t u|^2\,dx + \frac{d}{dt}\int_{\mathbb{R}^2} \mu(\theta)d : d\,dx &\le C\Big(\|u\cdot\nabla u\|_{L^2}^2 \\ &\quad + \big(\|\nabla\theta\|_{L^2} + \|\nabla u\|_{L^2}\big)\|\nabla u\|_{L^4}^2 + \|\theta\|_{L^2}^2\Big) - \int_{\mathbb{R}^2} u\cdot\nabla u \mid \nabla\Pi\,dx + \frac{1}{4}\|\partial_t u\|_{L^2}^2. \end{aligned}$$

Integrating the above inequality over $[0,t]$ and using once again $\operatorname{div} u = 0$, we infer

$$\begin{aligned}\frac{3}{4}\|\partial_t u\|_{L^2_t(L^2)}^2+\|\nabla u(t)\|_{L^2}^2 \leq C\Big(\|\nabla u_0\|_{L^2}^2+\|\theta\|_{L^2_t(L^2)}^2+\int_0^t \|u\cdot\nabla u\|_{L^2}^2\,dt' \\ +\int_0^t \big(\|\nabla\theta\|_{L^2}+\|\nabla u\|_{L^2}\big)\|\nabla u\|_{L^4}^2\,dt' + \sum_{i,k=1}^2\int_0^t\int_{\mathbb{R}^2}\Pi\partial_i u^k\partial_k u^i\,dx\,dt'\Big).\end{aligned} \tag{4.18}$$

To deal with the pressure function Π, we get, by taking space divergence to the momentum equation of (1.1), that

$$\Pi=(-\Delta)^{-1}\mathrm{div}\otimes\mathrm{div}\big(2\mu(\theta)d\big)-(-\Delta)^{-1}\mathrm{div}\big(u\cdot\nabla u-\theta e_2\big), \tag{4.19}$$

from which, we deduce

$$\begin{aligned}\Big|\sum_{i,k=1}^2\int_{\mathbb{R}^2}\Pi\partial_i u^k\partial_k u^i\,dx\Big| &\lesssim \|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2 \\ &+\|(-\Delta)^{-1}\mathrm{div}\big(u\cdot\nabla u-\theta e_2\big)\|_{BMO}\Big\|\sum_{i,k=1}^2\partial_i u^k\partial_k u^i\Big\|_{\mathscr{H}^1},\end{aligned}$$

where $\|f\|_{\mathscr{H}^1}$ denotes the Hardy norm of f. Yet as $\operatorname{div} u=0$, it follows from [11] that

$$\Big\|\sum_{i,k=1}^2\partial_i u^k\partial_k u^i\Big\|_{\mathscr{H}^1}\lesssim\|\nabla u\|_{L^2}^2,$$

and $\|f\|_{BMO(\mathbb{R}^2)}\lesssim\|\nabla f\|_{L^2(\mathbb{R}^2)}$, we obtain

$$\Big|\sum_{i,k=1}^2\int_{\mathbb{R}^2}\Pi\partial_i u^k\partial_k u^i\,dx\Big|\lesssim\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2+\|u\cdot\nabla u-\theta e_2\|_{L^2}\|\nabla u\|_{L^2}^2,$$

from which, and $\|u\|_{L^4}^2\lesssim\|u\|_{L^2}\|\nabla u\|_{L^2}$, we deduce from (4.18) that

$$\begin{aligned}\frac{3}{4}\|\partial_t u\|_{L^2_t(L^2)}^2+\|\nabla u(t)\|_{L^2}^2\leq C\Big(\|\nabla u_0\|_{L^2}^2+\|\theta\|_{L^2_t(L^2)}^2 \\ +\int_0^t\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2\,dt'+\int_0^t\big((1+\|u\|_{L^2})\|\nabla u\|_{L^2}+\|\nabla\theta\|_{L^2}\big)\|\nabla u\|_{L^4}^2\,dt'\Big).\end{aligned} \tag{4.20}$$

On the other hand, by virtue of (2.4), and

$$\|a\|_{L^p}\leq C\sqrt{p}\|a\|_{L^2}^{\frac{2}{p}}\|\nabla a\|_{L^2}^{1-\frac{2}{p}}\quad\text{for}\quad p\in[2,\infty[, \tag{4.21}$$

we infer

$$\|\nabla u\|_{L^p}\leq C\Big(p\|\mu(\theta)-1\|_{L^\infty}\|\nabla u\|_{L^p}+\sqrt{p}\|\nabla u\|_{L^2}^{\frac{2}{p}}\|\mathbb{P}\mathrm{div}\big(2\mu(\theta)d\big)\|_{L^2}^{1-\frac{2}{p}}\Big)$$

Taking ε_0 sufficiently small in (1.5), we obtain for $p\in\big[2,p^*\big]$

$$\begin{aligned}\|\nabla u\|_{L^p}&\leq C\sqrt{p}\|\nabla u\|_{L^2}^{\frac{2}{p}}\|\partial_t u+(u\cdot\nabla)u-\theta e_2\|_{L^2}^{1-\frac{2}{p}} \\ &\leq C\sqrt{p}\|\nabla u\|_{L^2}^{\frac{2}{p}}\Big(\|\partial_t u\|_{L^2}^{1-\frac{2}{p}}+\|u\|_{L^4}^{1-\frac{2}{p}}\|\nabla u\|_{L^4}^{1-\frac{2}{p}}+\|\theta\|_{L^2}^{1-\frac{2}{p}}\Big).\end{aligned} \tag{4.22}$$

In particular taking $p=4$ in (4.22) and using Young's inequality yields

$$\|\nabla u\|_{L^4}^2\leq C\Big(\|\nabla u\|_{L^2}\|\partial_t u\|_{L^2}+\|u\|_{L^2}\|\nabla u\|_{L^2}^3+\|\nabla u\|_{L^2}\|\theta\|_{L^2}\Big). \tag{4.23}$$

Substituting (4.23) into (4.20), we obtain

$$\begin{aligned}\|\partial_t u\|_{L^2_t(L^2)}^2 + \|\nabla u(t)\|_{L^2}^2 \leq C\Big(\|\nabla u_0\|_{L^2}^2 + \|\theta\|_{L^2_t(L^2)}^2 \\ + \int_0^t \big((1+\|u\|_{L^2}^2)\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\big)\|\nabla u\|_{L^2}^2\,dt'\Big).\end{aligned} \tag{4.24}$$

Note from (1.7) that

$$\|\theta\|_{L^2_t(L^2)} \lesssim E_0. \tag{4.25}$$

Applying Gronwall's Lemma to (4.24) gives

$$\begin{aligned}\|\nabla u\|_{L^\infty_t(L^2)}^2 + \|\partial_t u\|_{L^2_t(L^2)}^2 &\leq C\big(\|\nabla u_0\|_{L^2}^2 + E_0^2\big) \\ &\quad\times \exp\Big(C\int_0^t \big((1+\|u\|_{L^2}^2)\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\big)(t')\,dt'\Big) \\ &\leq C\big(E_0^2 + \|\nabla u_0\|_{L^2}^2\big)\exp\big(CE_0^2(1+E_0^2)\big)\exp\big(C\|\nabla\theta\|_{L^2_t(L^2)}^2\big).\end{aligned} \tag{4.26}$$

Substituting (4.11) into the above inequality leads to (4.15). This completes the proof of the lemma. $\square$

An immediate consequence of Lemma 4.3 is the following Corollary concerning the estimates of $\dot H^1$ energy estimate for u and of $\|\nabla u\|_{L^2_t(L^p)}$.

Corollary 4.1. *Under the assumptions of Lemma 4.3, we assume further that δ_1 given by (4.16) satisfies*

$$\delta_1 \leq 1. \tag{4.27}$$

Then for any $t \in]0, T^*[$, one has

$$\begin{aligned}&\|\nabla u\|_{L^\infty_t(L^2)}^2 + \|\partial_t u\|_{L^2_t(L^2)}^2 \leq C\mathfrak{C}_0\Big(1 + \|\theta_0\|_{B^{1/2}_{p,\infty}} + \mathfrak{C}_0 pE_0^2(1+E_0)^{2\left(1-\frac{2}{p}\right)}\|\theta_0\|_{L^\infty}^2\Big) \stackrel{def}{=} \mathfrak{C}_{p,1}^2, \\ &\|\nabla u\|_{L^2_t(L^p)} \leq C\sqrt{p}E_0(1+E_0)^{1-\frac{2}{p}}\big(1+\mathfrak{C}_{p,1}\big).\end{aligned} \tag{4.28}$$

Proof. We first deduce from (2.1), (4.23) and (4.25) that

$$\begin{aligned}\|\nabla u\|_{L^2_t(L^4)} \leq& C\Big(\|\nabla u\|_{L^2_t(L^2)}^{\frac12}\|\partial_t u\|_{L^2_t(L^2)}^{\frac12} + \|u\|_{L^\infty_t(L^2)}^{\frac12}\|\nabla u\|_{L^\infty_t(L^2)}^{\frac12}\|\nabla u\|_{L^2_t(L^2)} \\ &+ \|\nabla u\|_{L^2_t(L^2)}^{\frac12}\|\theta\|_{L^2_t(L^2)}^{\frac12}\Big) \\ \leq& C(1+E_0)E_0^{\frac12}\Big(1 + \|\partial_t u\|_{L^2_t(L^2)}^{\frac12} + \|\nabla u\|_{L^\infty_t(L^2)}^{\frac12}\Big),\end{aligned} \tag{4.29}$$

from which and (4.22), we infer

$$\begin{aligned}\|\nabla u\|_{L^2_t(L^p)} \leq& C\sqrt{p}\|\nabla u\|_{L^2_t(L^2)}^{\frac2p}\Big(\|\partial_t u\|_{L^2_t(L^2)}^{1-\frac2p} + \|u\|_{L^\infty_t(L^4)}^{1-\frac2p}\|\nabla u\|_{L^2_t(L^4)}^{1-\frac2p} + \|\theta\|_{L^2_t(L^2)}^{1-\frac2p}\Big) \\ \leq& C\sqrt{p}E_0(1+E_0)^{1-\frac2p}\big(1 + \|\partial_t u\|_{L^2_t(L^2)} + \|\nabla u\|_{L^\infty_t(L^2)}\big).\end{aligned} \tag{4.30}$$

Resuming the Estimate (4.30) into (4.15), we write

$$\begin{aligned}\|\nabla u\|_{L^\infty_t(L^2)}^2 + \|\partial_t u\|_{L^2_t(L^2)}^2 \leq C\mathfrak{C}_0\Big(1 + \|\theta_0\|_{B^{1/2}_{p,\infty}} \\ + \sqrt{p}E_0(1+E_0)^{1-\frac2p}\|\theta_0\|_{L^\infty}\big(1 + \|\partial_t u\|_{L^2_t(L^2)} + \|\nabla u\|_{L^\infty_t(L^2)}\big)\Big)^{\delta_1}.\end{aligned} \tag{4.31}$$

In particular, under the assumption of (4.27), we deduce the first inequality of (4.28) from (4.31).

Substituting the first inequality of (4.28) into (4.30), we obtain the second inequality of (4.28). This completes the proof of the Corollary. $\square$

To prove the general global in time *a priori* $\dot{H}^1$ energy estimate for u without the restriction (4.27), we need the following lemma concerning the non-concentration of energy in the time variable:

Lemma 4.4. *Let (θ, u) be a smooth enough solution of the System (1.1) on $[0, T^*[$. Then under the assumptions of Proposition 4.1, one has for all $t \in [0, T^*[$*

$$\|u\|_{\widetilde{L}^\infty_t(L^2)} \leq CE_0(1+E_0) \tag{4.32}$$

for E_0 given by (1.8). If moreover, there holds (1.5), then for any small enough constant $\kappa > 0$, there exists $\lambda > 0$ such that if $0 \leq t_1 < t_2 < T^*$ and $t_2 - t_1 \leq \lambda$, there holds

$$\|\nabla u\|_{L^2([t_1,t_2];L^2)} \leq \kappa. \tag{4.33}$$

Proof. We first get, by applying $\dot{\Delta}_j$ to the u equation of (1.1), that

$$\partial_t \dot{\Delta}_j u + (u\cdot\nabla)\dot{\Delta}_j u - \Delta\dot{\Delta}_j u + \nabla\dot{\Delta}_j\Pi = [u\cdot\nabla, \dot{\Delta}_j]u + \mathrm{div}\dot{\Delta}_j\big((\mu(\theta)-1)2d\big) + \dot{\Delta}_j\theta e_2. \tag{4.34}$$

Taking L^2 inner product of the previous equation with $\dot{\Delta}_j u$ and using Lemma 3.1, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\dot{\Delta}_j u\|_{L^2}^2 + c2^{2j}\|\dot{\Delta}_j u\|_{L^2}^2 \leq \|\dot{\Delta}_j u\|_{L^2}\\ \times\Big(\|[\dot{\Delta}_j; u\cdot\nabla]u\|_{L^2} + C2^j\|\dot{\Delta}_j\big((\mu(\theta)-1)\nabla u\big)\|_{L^2} + \|\dot{\Delta}_j\theta\|_{L^2}\Big),\end{aligned} \tag{4.35}$$

which gives rise to

$$\begin{aligned}\|\dot{\Delta}_j u(t)\|_{L^2} \lesssim & e^{-c2^{2j}t}\|\dot{\Delta}_j u_0\|_{L^2} + \int_0^t e^{-c2^{2j}(t-t')}\big(\|[\dot{\Delta}_j; u\cdot\nabla]u\|_{L^2}\\ &+ \|\dot{\Delta}_j\theta\|_{L^2}\big)(t')\,dt' + C2^j\int_0^t e^{-c2^{2j}(t-t')}\|\dot{\Delta}_j\big((\mu(\theta)-1)\nabla u\big)(t')\|_{L^2}\,dt'.\end{aligned}$$

Taking $L^\infty(0,t)$ norm with respect to time t and then taking $\ell^2(\mathbb{Z})$ norm to the resulting inequality, we get

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(L^2)} \lesssim & \|u_0\|_{L^2} + \Big(\sum_{j\in\mathbb{Z}}\|[\dot{\Delta}_j; u\cdot\nabla]u\|_{L^1_t(L^2)}^2\Big)^{\frac{1}{2}}\\ &+ \|(\mu(\theta)-1)\nabla u\|_{L^2_t(L^2)} + \Big(\sum_{j\in\mathbb{Z}}\|\dot{\Delta}_j\theta\|_{L^1_t(L^2)}^2\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.36}$$

Note from (1.4) and (2.1) that

$$\|(\mu(\theta)-1)\nabla u\|_{L^2_t(L^2)} \leq \|(\mu(\theta)-1)\|_{L^\infty_t(L^\infty)}\|\nabla u\|_{L^2_t(L^2)} \leq CE_0.$$

While it follows from Minkowski inequality and the Inequality (4.9) that

$$\Big(\sum_{j\in\mathbb{Z}}\|\dot{\Delta}_j\theta\|_{L^1_t(L^2)}^2\Big)^{\frac{1}{2}} \leq \|\theta\|_{L^1_t(L^2)} \leq CE_0.$$

Finally similar to the proof of Lemme A.1 of [14], we use Bony's decomposition to write

$$[\dot{\Delta}_j; u\cdot\nabla]u = [\dot{\Delta}_j; T_u\cdot\nabla]u + \dot{\Delta}_j T_{\nabla u}u - T_{\dot{\Delta}_j\nabla u}u + \dot{\Delta}_j R(u,\nabla u) - R(u, \dot{\Delta}_j\nabla u).$$

Applying commutator's estimate ([7]) gives

$$\begin{aligned}\|[\dot{\Delta}_j; T_u\cdot\nabla]u\|_{L^1_t(L^2)} \lesssim & 2^{-j}\sum_{|j'-j|\leq 4}\|S_{j'-1}\nabla u\|_{L^2_t(L^\infty)}\|\dot{\Delta}_{j'}\nabla u\|_{L^2_t(L^2)}\\ \lesssim & \sum_{|j'-j|\leq 4} c_j\|\nabla u\|_{L^2_t(L^2)}\|\dot{\Delta}_{j'}\nabla u\|_{L^2_t(L^2)}\\ \lesssim & d_j\|\nabla u\|_{L^2_t(L^2)}^2.\end{aligned}$$

Similarly, one has

$$\begin{aligned}\|\dot\Delta_j T_{\nabla u}u\|_{L^2_t(L^2)} &\lesssim \sum_{|j'-j|\le 4}\|S_{j'-1}\nabla u\|_{L^2_t(L^\infty)}\|\dot\Delta_{j'}u\|_{L^2_t(L^2)}\\ &\lesssim d_j\|\nabla u\|^2_{L^2_t(L^2)}.\end{aligned}$$

The same estimate holds for $T_{\dot\Delta_j\nabla u}u$. While since $\operatorname{div} u=0$, applying Lemma 3.1 yields

$$\begin{aligned}\|\dot\Delta_j R(u,\nabla u)\|_{L^2_t(L^2)} &\lesssim \|\operatorname{div}\dot\Delta_j R(u,u)\|_{L^2_t(L^2)}\\ &\lesssim 2^{2j}\sum_{j'\ge j-3}\|\dot\Delta_j u\|_{L^2_t(L^2)}\|\tilde{\dot\Delta}_j u\|_{L^2_t(L^2)}\lesssim d_j\|\nabla u\|^2_{L^2_t(L^2)}.\end{aligned}$$

The same estimate holds for $R(u,\dot\Delta_j\nabla u)$. We thus obtain

$$\Big(\sum_q\|[\dot\Delta_j;u\cdot\nabla]u\|^2_{L^1_t(L^2)}\Big)^{\frac12}\lesssim\|\nabla u\|^2_{L^2_t(L^2)}. \tag{4.37}$$

Resuming the above estimates into (4.36) and using (2.1) leads to (4.32).

Along the same line, for any $0\le t_1<t_2<T^*$, we deduce from (4.35) that

$$\begin{aligned}\|\nabla u\|_{L^2(t_1,t_2;L^2)} \lesssim&\Big(\sum_{j\in\mathbb Z}\big(1-e^{-2c2^{2j}(t_2-t_1)}\big)\|\dot\Delta_j u(t_1)\|^2_{L^2}\Big)^{\frac12}+\Big(\sum_{j\in\mathbb Z}\|\dot\Delta_j\theta\|^2_{L^1(t_1,t_2;L^2)}\Big)^{\frac12}\\ &+\Big(\sum_{j\in\mathbb Z}\|[\dot\Delta_j;u\cdot\nabla]u\|^2_{L^1(t_1,t_2;L^2)}\Big)^{\frac12}+\|\big(\mu(\theta)-1\big)\nabla u\|_{L^2(t_1,t_2;L^2)},\end{aligned}$$

from which, and (4.37), we infer

$$\begin{aligned}\|\nabla u\|_{L^2(t_1,t_2;L^2)} \lesssim&\Big(\sum_{j\in\mathbb Z}\big(1-e^{-2c2^{2j}(t_2-t_1)}\big)\|\dot\Delta_j u(t_1)\|^2_{L^2}\Big)^{\frac12}+\|\theta\|_{L^1(t_1,t_2;L^2)}\\ &+\|\mu(\theta)-1\|_{L^\infty}\|\nabla u\|_{L^2(t_1,t_2;L^2)}+\|\nabla u\|^2_{L^2(t_1,t_2;L^2)}.\end{aligned}$$

So that under the assumption of (1.5), one has

$$\begin{aligned}\|\nabla u\|_{L^2(t_1,t_2;L^2)} \lesssim&\Big(\sum_{j\in\mathbb Z}\big(1-e^{-2c2^{2j}(t_2-t_1)}\big)\|\dot\Delta_j u(t_1)\|^2_{L^2}\Big)^{\frac12}\\ &+\|\nabla u\|^2_{L^2(t_1,t_2;L^2)}+(t_2-t_1)\|\theta_0\|_{L^2}.\end{aligned} \tag{4.38}$$

Note that for any integer N, one has

$$\begin{aligned}\Big(\sum_{j\in\mathbb Z}\big(1-e^{-2c2^{2j}(t_2-t_1)}\big)\|\dot\Delta_j u(t_1)\|^2_{L^2}\Big)^{\frac12}\le&\Big(\sum_{|j|\ge N}\|\dot\Delta_j u(t_1)\|^2_{L^2}\Big)^{\frac12}\\ &+\Big(\sum_{|j|<N}\big(1-e^{-2c2^{2j}(t_2-t_1)}\big)\|\dot\Delta_j u(t_1)\|^2_{L^2}\Big)^{\frac12}.\end{aligned}$$

Yet by virtue of (4.32), for any $\eta>0$, there exists some N_0 so that there holds

$$\Big(\sum_{|j|\ge N_0}\|\dot\Delta_j u(t_1)\|^2_{L^2}\Big)^{\frac12}\le\Big(\sum_{|j|\ge N_0}\|\dot\Delta_j u\|^2_{L^\infty(\mathbb R_+;L^2)}\Big)^{\frac12}\le\frac{\eta}{3}$$

and

$$\Big(\sum_{|j|<N_0}\big(1-e^{-2c2^{2j}(t_2-t_1)}\big)\|\dot\Delta_q u(t_1)\|^2_{L^2}\Big)^{\frac12}\le CE_0(1+E_0)\sqrt{t_2-t_1}\,2^{N_0}.$$

If λ is taken so small that $CE_0(1+E_0)\sqrt{\lambda}2^{N_0} \leq \frac{\eta}{3}$ and $C\lambda\|\theta_0\|_{L^2} \leq \frac{\eta}{3}$, then we infer from (4.38) that

$$\|\nabla u\|_{L^2(t_1,t_2;L^2)} \leq \eta + C\|\nabla u\|^2_{L^2(t_1,t_2;L^2)},$$

which implies

$$\begin{aligned}&\text{either } \|\nabla u\|_{L^2(t_1,t_2;L^2)} \geq \frac{1+\sqrt{1-4C\eta}}{2C}\\ &\text{or } \quad \|\nabla u\|_{L^2(t_1,t_2;L^2)} \leq \frac{1-\sqrt{1-4C\eta}}{2C} = \frac{2\eta}{1+\sqrt{1-4C\eta}} \leq 2\eta,\end{aligned}$$

provided that η is so small that $C\eta \leq \frac{1}{4}$. Since when $t_2 = t_1$, one has $\|\nabla u\|_{L^2(t_1,t_2;L^2)} = 0$, we get, by applying the absolute continuity of L^1 functions, that

$$\|\nabla u\|_{L^2([t_1,t_2;L^2)} \leq 2\eta,$$

which implies (4.33). This concludes the proof of Lemma 4.4. $\square$

Now let us turn to the key estimates in this subsection:

Proposition 4.2. *Let (θ, u) be a smooth enough solution of the System (1.1) on $[0,T^*[$. Then under the assumptions of Proposition 4.1 and (1.5) for some sufficiently small ε_0, for any $t < T^*$ and $p \in [4, p^*]$, one has*

$$\begin{aligned}&\|\theta\|^2_{\widetilde{L}^\infty_t(\dot{H}^{1/2})} + \ln\big(1+\|\theta\|^2_{L^\infty_t(B^{1/2}_{p,\infty})}\big) + \|\theta\|^2_{L^2_t(\dot{H}^1)}\\ &\quad \leq C^{2+C\|\theta_0\|^2_{L^\infty}E_0^2}\Big(\mathcal{A}+\mathcal{B}+\|\theta_0\|^2_{\dot{H}^{1/2}} + \ln\big(1+\|\theta_0\|^2_{B^{1/2}_{p,\infty}} + \|\nabla u_0\|^2_{L^2}\big)\Big) \stackrel{\text{def}}{=} \mathcal{G}_1,\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}\|\nabla u\|^2_{L^\infty_t(L^2)} + \|\partial_t u\|^2_{L^2_t(L^2)} \leq & C\big(1+\|\nabla u_0\|^2_{L^2}\big)\exp\Big(C\big(E_0^2(1+E_0^2)+\mathcal{G}_1\big)\Big) \stackrel{\text{def}}{=} \mathcal{G}_2,\\ \|\nabla u\|_{L^2_t(L^p)} \leq & CE_0(1+E_0)^{1-\frac{2}{p}}\big(1+\sqrt{\mathcal{G}_2}\big) \stackrel{\text{def}}{=} \mathcal{G}_3,\end{aligned} \tag{4.40}$$

where the constants $\mathcal{A}, \mathcal{B}$ are defined by

$$\begin{aligned}\mathcal{A} \stackrel{\text{def}}{=} & CE_0^2\big(1+E_0^2+\|\theta_0\|^2_{L^2\cap L^\infty}\big),\\ \mathcal{B} \stackrel{\text{def}}{=} & \mathcal{A} + CE_0^2\|\theta_0\|^2_{L^\infty}\ln\big(1+CE_0(1+E_0)\|\theta_0\|_{L^\infty}\big).\end{aligned} \tag{4.41}$$

Proof. Let us first consider any subinterval $I = [I^-, I^+]$ of $[0,T^*[$. Then a similar proof of (4.26) ensures (2.2). While it follows from Lemma 4.1 that

$$\|\theta\|_{L^\infty_t(L^\infty)} \leq \|\theta_0\|_{L^\infty},$$

so that we deduce by a similar proof of (4.13) that

$$\begin{aligned}\|\theta\|^2_{\widetilde{L}^\infty(I;\dot{H}^{1/2})} + \|\theta\|^2_{L^2(I;\dot{H}^1)} \leq & \|\theta(I^-)\|^2_{\dot{H}^{1/2}} + C\|\nabla u\|^2_{L^2(I;L^2)}\big(\|\theta\|^2_{L^\infty(I;L^\infty)} + \|\theta\|^2_{L^\infty(I;\dot{B}^0_{\infty,2})}\big)\\ \leq & \|\theta(I^-)\|^2_{\dot{H}^{1/2}} + C\|\nabla u\|^2_{L^2(I;L^2)}\big(\|\theta_0\|^2_{L^\infty} + \|\theta\|^2_{L^\infty(I;\dot{B}^0_{\infty,2})}\big),\end{aligned}$$

which together with (4.14) implies (2.3). Resuming (2.3) into (2.2) and using (2.1), we obtain

$$\begin{aligned}&\|\nabla u\|^2_{L^\infty(I;L^2)} + \|\partial_t u\|^2_{L^2(I;L^2)}\\ &\qquad \leq CA(I^-)\big(1+\|\theta(I^-)\|_{B^{1/2}_{p,\infty}} + \|\theta_0\|_{L^\infty}\|\nabla u\|_{L^2(I;L^p)}\big)^{C\|\theta_0\|^2_{L^\infty}\|\nabla u\|^2_{L^2(I;L^2)}},\end{aligned}$$

where

$$A(I^-) \stackrel{\text{def}}{=} \big(1+\|\nabla u(I^-)\|^2_{L^2}\big)\exp\Big(CE_0^2\big(1+E_0^2+\|\theta_0\|^2_{L^2\cap L^\infty}\big) + C\|\theta(I^-)\|^2_{\dot{H}^{1/2}}\Big). \tag{4.42}$$

On the other hand, we deduce from the proof of (4.30) that there holds (2.5). Whence we obtain (2.6). However, by virtue of Lemma 4.4, we can take $|I| = I^+ - I^-$ to be so small that

$$C\|\theta_0\|_{L^\infty}^2\|\nabla u\|_{L^2(I;L^2)}^2 \leq 1.$$

Then we infer from (2.6) that

$$\begin{aligned}
&\|\nabla u\|_{L^\infty(I;L^2)}^2 + \|\partial_t u\|_{L^2(I;L^2)}^2 \\
&\qquad\leq C\big(1 + \|\theta(I^-)\|_{B^{1/2}_{p,\infty}}^2 + (1 + E_0^2(1+E_0^2)\|\theta_0\|_{L^\infty}^2)A^2(I^-)\big) \\
&\qquad\leq C\Big(1 + \|\theta(I^-)\|_{B^{1/2}_{p,\infty}}^2 + \big(1 + \|\nabla u(I^-)\|_{L^2}^2\big)^2 \\
&\qquad\qquad \times \exp\Big(2CE_0^2\big(1 + E_0^2 + \|\theta_0\|_{L^2\cap L^\infty}^2\big) + 2C\|\theta(I^-)\|_{\dot H^{1/2}}^2\Big)\Big).
\end{aligned} \tag{4.43}$$

Taking ln to (4.43) yields

$$\begin{aligned}
&\ln\big(1 + \|\nabla u\|_{L^\infty(I;L^2)}^2 + \|\partial_t u\|_{L^2(I;L^2)}^2\big) \\
&\qquad\leq \frac{1}{2}\mathcal{A} + C\Big(\|\theta(I^-)\|_{\dot H^{1/2}}^2 + \ln\big(1 + \|\theta(I^-)\|_{B^{1/2}_{p,\infty}}^2 + \|\nabla u(I^-)\|_{L^2}^2\big)\Big),
\end{aligned} \tag{4.44}$$

for $\mathcal{A}$ given by (4.41).

Whereas substituting the Estimate (4.43) into (2.5) leads to

$$\begin{aligned}
\|\nabla u\|_{L^2(I;L^p)} \leq &CE_0(1+E_0)\big(1 + \|\theta(I^-)\|_{B^{1/2}_{p,\infty}}\big) + C\big(1 + \|\nabla u(I^-)\|_{L^2}^2\big) \\
&\times \exp\Big(CE_0^2\big(1 + E_0^2 + \|\theta_0\|_{L^2\cap L^\infty}^2\big) + C\|\theta(I^-)\|_{\dot H^{1/2}}^2\Big),
\end{aligned} \tag{4.45}$$

from which and (2.3), we infer

$$\|\theta\|_{\widetilde L^\infty(I;\dot H^{1/2})}^2 + \|\theta\|_{L^2(I;\dot H^1)}^2 \leq \mathcal{B} + C\Big(\|\theta(I^-)\|_{\dot H^{1/2}}^2 + \ln\big(1 + \|\theta(I^-)\|_{B^{1/2}_{p,\infty}}^2 + \|\nabla u(I^-)\|_{L^2}^2\big)\Big), \tag{4.46}$$

for $\mathcal{B}$ given by (4.41).

By the same manner, we infer from (4.10) and (4.45) that

$$\ln\big(1 + \|\theta\|_{L^\infty(I;B^{1/2}_{p,\infty})}^2\big) \leq \frac{1}{2}\mathcal{A} + C\Big(\|\theta(I^-)\|_{\dot H^{1/2}}^2 + \ln\big(1 + \|\theta(I^-)\|_{B^{1/2}_{p,\infty}}^2 + \|\nabla u(I^-)\|_{L^2}^2\big)\Big). \tag{4.47}$$

Now for any $t \in]0, T^*[$, let us decompose $[0,t]$ into the intervals: $I_i = [t_i, t_{i+1}], i = 0, \cdots, N-1$ so that $[0,T] = \cup_{i=0}^{N-1}[t_i, t_{i+1}]$ with $t_0 = 0, t_N = t$ and

$$\forall i \in \{0, 1, \cdots, N-2\}, \quad C\|\theta_0\|_{L^\infty}^2\|\nabla u\|_{L^2(I_i;L^2)}^2 = 1 \quad \text{and} \quad C\|\theta_0\|_{L^\infty}^2\|\nabla u\|_{L^2(I_{N-1};L^2)}^2 \leq 1.$$

Let us observe from (2.1) that

$$N \leq 1 + C\|\theta_0\|_{L^\infty}^2\|\nabla u\|_{L^2_t(L^2)}^2 \leq 1 + CE_0^2\|\theta_0\|_{L^\infty}^2. \tag{4.48}$$

It follows from (4.44), (4.46) and (4.47) that

$$\begin{aligned}
f_i \stackrel{\text{def}}{=} &\|\theta\|_{\widetilde L^\infty(I_i;\dot H^{1/2})}^2 + \|\theta\|_{L^2(I_i;\dot H^1)}^2 \\
&\quad + \ln\big(1 + \|\theta\|_{L^\infty(I_i;B^{1/2}_{p,\infty})}^2 + \|\nabla u\|_{L^\infty(I_i;L^2)}^2 + \|\partial_t u\|_{L^2(I_i;L^2)}^2\big) \\
\leq &\|\theta\|_{\widetilde L^\infty(I_i;\dot H^{1/2})}^2 + \|\theta\|_{L^2(I_i;\dot H^1)}^2 + \ln\big(1 + \|\theta\|_{L^\infty(I_i;B^{1/2}_{p,\infty})}^2\big) \\
&\quad + \ln\big(1 + \|\nabla u\|_{L^\infty(I_i;L^2)}^2 + \|\partial_t u\|_{L^2(I_i;L^2)}^2\big) \\
\leq &\mathcal{A} + \mathcal{B} + C\Big(\|\theta(t_i)\|_{\dot H^{1/2}}^2 + \ln\big(1 + \|\theta(t_i)\|_{B^{1/2}_{p,\infty}}^2 + \|\nabla u(t_i)\|_{L^2}^2\big)\Big) \\
\leq &\mathcal{A} + \mathcal{B} + Cf_{i-1}.
\end{aligned} \tag{4.49}$$

Inductively, we deduce from (4.49) that for $i \geq 2$

$$f_i \leq (\mathcal{A}+\mathcal{B})\frac{C^{i-1}-1}{C-1}+C^{i-1}f_1.$$

Yet observe from (4.49) that

$$f_1 \leq \mathcal{A}+\mathcal{B}+C\Big(\|\theta_0\|_{\dot H^{1/2}}^2+\ln\big(1+\|\theta_0\|_{B^{1/2}_{p,\infty}}^2+\|\nabla u_0\|_{L^2}^2\big)\Big).$$

We thus obtain

$$f_i \leq (\mathcal{A}+\mathcal{B})\frac{C^{i}-1}{C-1}+C^{i}\Big(\|\theta_0\|_{\dot H^{1/2}}^2+\ln\big(1+\|\theta_0\|_{B^{1/2}_{p,\infty}}^2+\|\nabla u_0\|_{L^2}^2\big)\Big), \tag{4.50}$$

which together with (4.48) and (4.49) implies

$$\begin{aligned}\|\theta\|^2_{\widetilde L^\infty_t(\dot H^{1/2})}&+\ln\big(1+\|\theta\|^2_{L^\infty_t(B^{1/2}_{p,\infty})}\big)\\ &\leq C^{1+C\|\theta_0\|^2_{L^\infty}E_0^2}\Big(\mathcal{A}+\mathcal{B}+\|\theta_0\|_{\dot H^{1/2}}^2+\ln\big(1+\|\theta_0\|_{B^{1/2}_{p,\infty}}^2+\|\nabla u_0\|_{L^2}^2\big)\Big).\end{aligned} \tag{4.51}$$

Then (4.49) and (4.50) ensures that

$$\|\theta\|^2_{L^2_t(\dot H^1)} \leq C^{2+C\|\theta_0\|^2_{L^\infty}E_0^2}\Big(\mathcal{A}+\mathcal{B}+\|\theta_0\|_{\dot H^{1/2}}^2+\ln\big(1+\|\theta_0\|_{B^{1/2}_{p,\infty}}^2+\|\nabla u_0\|_{L^2}^2\big)\Big),$$

which together with the (4.51) leads to (4.39). Substituting (4.39) into (4.26) gives the first inequality of (4.40). Finally substituting the first inequality of (4.40) into (4.30) leads to the second inequality of (4.40). This completes the proof of Proposition 4.2. ☐

4.3. **The improved derivative estimates of** (θ,u)**.** Based on the energy estimates of (θ,u) in the last subsection, we shall present the estimates of $\|u\|_{L^1_t(\dot B^1_{\infty,1})}$ and $\|\theta\|_{\widetilde L^1_t(\dot B^{3/2}_{p,\infty})}$ in this subsection.

Proposition 4.3. *Let (θ,u) be smooth enough solution of the System (1.1) on $[0,T^*[$. Then under the assumptions of Proposition 4.2, for any $t<T^*$, if we assume moreover that $u_0\in \dot B^{-1}_{\infty,1}$, there holds*

$$\begin{aligned}\|u\|_{L^1_t(\dot B^1_{\infty,1})}\leq C\Big(&\|u_0\|_{\dot B^{-1}_{\infty,1}}+E_0^2(1+E_0)+t\|\theta_0\|_{L^q}^{\frac{q}{2}}\|\theta_0\|_{L^\infty}^{1-\frac{q}{2}}\\ &+\mathcal{G}_3\mathcal{G}_4^{\frac{p+2}{2(p-1)}}\big(\|\theta_0\|_{L^2}\sqrt{t}\big)^{\frac{p-4}{2(p-1)}}\Big)\stackrel{\text{def}}{=}\mathcal{H}(t),\end{aligned} \tag{4.52}$$

for $\mathcal{G}_3$ given by (4.40) and $\mathcal{G}_4\stackrel{\text{def}}{=}\|\theta_0\|_{B^{1/2}_{p,\infty}}+C\|\theta_0\|_{L^\infty}\mathcal{G}_3$.

Proof. We first deduce from (4.10) and (4.40) that

$$\|\theta\|_{\widetilde L^\infty_t(B^{1/2}_{p,\infty})}+\|\theta\|_{\widetilde L^2_t(B^1_{p,\infty})}\leq \|\theta_0\|_{B^{1/2}_{p,\infty}}+C\|\theta_0\|_{L^\infty}\mathcal{G}_3\stackrel{\text{def}}{=}\mathcal{G}_4. \tag{4.53}$$

While we deduce from (4.34) that

$$\dot\Delta_j u(t)=e^{t\Delta}\dot\Delta_j u_0+\int_0^t e^{(t-t')\Delta}\dot\Delta_j\mathbb{P}\big(-u\cdot\nabla u+2\mathrm{div}\big((\mu(\theta)-1)d\big)+\theta e_2\big)(t')\,dt',$$

from which and Definition 3.2, we infer

$$\|u\|_{L^1_t(\dot B^1_{\infty,1})}\lesssim\|u_0\|_{\dot B^{-1}_{\infty,1}}+\|u\cdot\nabla u\|_{L^1_t(\dot B^{-1}_{\infty,1})}+\|(\mu(\theta)-1)\nabla u\|_{L^1_t(\dot B^{0}_{\infty,1})}+\|\theta\|_{L^1_t(\dot B^{-1}_{\infty,1})}. \tag{4.54}$$

In view of Bony's decomposition (3.5), we write

$$u\cdot\nabla u=T_u\nabla u+T_{\nabla u}u+R(u,\nabla u).$$

Applying Lemma 3.1 yields

$$\begin{aligned}\|\dot\Delta_j T_u\nabla u(t)\|_{L^\infty} &\lesssim \sum_{|j'-j|\leq 4}\|S_{j'-1}u(t)\|_{L^\infty}\|\dot\Delta_{j'}\nabla u(t)\|_{L^\infty}\\ &\lesssim \sum_{|j'-j|\leq 4} d_{j'}(t)2^{j'}\|u(t)\|_{\dot H^{1/2}}\|\nabla u(t)\|_{L^2}^{1/2}\|u(t)\|_{\dot B^1_{\infty,1}}^{1/2}\\ &\lesssim d_j(t)2^j\|u(t)\|_{\dot H^{1/2}}\|\nabla u(t)\|_{L^2}^{\frac12}\|u(t)\|_{\dot B^1_{\infty,1}}^{\frac12}.\end{aligned}$$

While due to $\operatorname{div} u=0$, we get, by applying Lemma 3.1, that

$$\begin{aligned}\|\dot\Delta_j R(u,\nabla u)(t)\|_{L^\infty} &\lesssim 2^{3j}\sum_{j'\geq j-3}\|\dot\Delta_{j'}u(t)\|_{L^2}\|\dot{\widetilde\Delta}_{j'}u(t)\|_{L^2}\\ &\lesssim d_j(t)2^j\|\nabla u(t)\|_{L^2}^2.\end{aligned}$$

The same estimate holds for $\dot\Delta_j T_{\nabla u}u$. Hence we obtain

$$\|u\cdot\nabla u\|_{L^1_t(\dot B^{-1}_{\infty,1})}\leq C\Big(\|\nabla u\|_{L^2_t(L^2)}^2+\|u\|_{L^\infty_t(L^2)}^{\frac12}\|\nabla u\|_{L^2_t(L^2)}\|u\|_{L^1_t(\dot B^1_{\infty,1})}^{\frac12}\Big). \tag{4.55}$$

Note that since $q<2$, for any integer L, we have

$$\begin{aligned}\|\theta\|_{L^1_t(\dot B^{-1}_{\infty,1})} &\lesssim \sum_{j\leq L}2^{j\left(\frac2q-1\right)}\|\dot\Delta_j\theta\|_{L^1_t(L^q)}+\sum_{j\geq N}2^{-j}\|\dot\Delta_j\theta\|_{L^1_t(L^\infty)}\\ &\lesssim t\Big(\|\theta_0\|_{L^q}2^{L\left(\frac2q-1\right)}+\|\theta_0\|_{L^\infty}2^{-L}\Big),\end{aligned}$$

taking L in the above inequality so that

$$2^{\frac{2L}{q}}\sim\|\theta_0\|_{L^\infty}\Big/\|\theta_0\|_{L^q}$$

we obtain

$$\|\theta\|_{L^1_t(\dot B^{-1}_{\infty,1})}\leq Ct\|\theta_0\|_{L^q}^{\frac q2}\|\theta_0\|_{L^\infty}^{1-\frac q2}. \tag{4.56}$$

Along the same line, for $p>4$ and any integer N, we have

$$\begin{aligned}\|\theta\|_{\widetilde L^2_t(\dot B^{4/p}_{p,1})} &\leq \sum_{j\leq N}2^{j\left(1+\frac2p\right)}\|\dot\Delta_j\theta\|_{L^2_t(L^2)}+\sum_{j>N}2^{j\left(\frac4p-1\right)}\|\theta\|_{\widetilde L^2_t(\dot B^1_{p,\infty})}\\ &\leq C\Big(\sqrt t\|\theta_0\|_{L^2}2^{N\left(1+\frac2p\right)}+\|\theta\|_{\widetilde L^2_t(\dot B^1_{p,\infty})}2^{-N\left(1-\frac4p\right)}\Big),\end{aligned}$$

taking N in the above inequality so that

$$2^{2N\left(1-\frac1p\right)}\sim\|\theta\|_{\widetilde L^2_t(\dot B^1_{p,\infty})}\Big/\sqrt t\|\theta_0\|_{L^2},$$

which together with (4.53) leads to

$$\|\theta\|_{\widetilde L^2_t(\dot B^{\frac4p}_{p,1})}\leq C\mathcal G_4^{\frac{p+2}{2(p-1)}}\big(\|\theta_0\|_{L^2}\sqrt t\big)^{\frac{p-4}{2(p-1)}}. \tag{4.57}$$

On the other hand, by using Bony's decomposition (3.5) once again, we write

$$\big(\mu(\theta)-1\big)d=T_{\mu(\theta)-1}d+T_d\big(\mu(\theta)-1\big)+R(\mu(\theta)-1,d).$$

Applying Lemma 3.1 gives

$$\begin{aligned}\|\dot\Delta_j R(\mu(\theta)-1,d)\|_{L^1_t(L^\infty)} \lesssim & 2^{\frac{4j}{p}}\sum_{j'\geq j-3}\|\dot\Delta_{j'}(\mu(\theta)-1)\|_{L^2_t(L^p)}\|\dot{\widetilde\Delta}_{j'}\nabla u\|_{L^2_t(L^p)}\\ \lesssim & d_j\|\nabla u\|_{L^2_t(L^p)}\|\theta\|_{\widetilde L^2_t(\dot B^{\frac4p}_{p,1})},\end{aligned}$$

from which, (4.40) and (4.57), we deduce

$$\|\dot\Delta_j R(\mu(\theta)-1,d)\|_{L^1_t(L^\infty)} \leq C d_j \mathcal{G}_3\mathcal{G}_4^{\frac{p+2}{2(p-1)}}\big(\|\theta_0\|_{L^2}\sqrt t\big)^{\frac{p-4}{2(p-1)}}.$$

The same estimate holds for $T_d\big(\mu(\theta)-1\big)$. We thus obtain

$$\|(\mu(\theta)-1)d\|_{L^1_t(\dot B^0_{\infty,1})} \leq C\Big(\|\mu(\theta)-1\|_{L^\infty}\|u\|_{L^1_t(\dot B^1_{\infty,1})} + \mathcal{G}_3\mathcal{G}_4^{\frac{p+2}{2(p-1)}}\big(\|\theta_0\|_{L^2}\sqrt t\big)^{\frac{p-4}{2(p-1)}}\Big).$$

Resuming the above inequality, (4.55), (4.56) into (4.54) results in

$$\begin{aligned}\|u\|_{L^1_t(\dot B^1_{\infty,1})} \leq \|u_0\|_{\dot B^{-1}_{\infty,1}} + C\Big(&\|u\|^{\frac12}_{L^\infty_t(L^2)}\|\nabla u\|_{L^2_t(L^2)}\|u\|^{\frac12}_{L^1_t(\dot B^1_{\infty,1})}\\ &+\|\nabla u\|^2_{L^2_t(L^2)} + t\|\theta_0\|^{\frac q2}_{L^q}\|\theta_0\|^{1-\frac q2}_{L^\infty} + \mathcal{G}_3\mathcal{G}_4^{\frac{p+2}{2(p-1)}}\big(\|\theta_0\|_{L^2}\sqrt t\big)^{\frac{p-4}{2(p-1)}}\Big),\end{aligned}$$

from which, and (2.1), we infer (4.52). This completes the proof of Proposition 4.3. □

Corollary 4.2. *Under the assumption of Proposition 4.3, one has*

$$\|\theta\|_{\widetilde L^\infty_t(\dot B^{1/2}_{p,\infty})} + \|\theta\|_{\widetilde L^1_t(\dot B^{3/2}_{p,\infty})} \leq \|\theta_0\|_{\dot B^{1/2}_{p,\infty}}\exp\big(C\mathcal{H}(t)\big) \tag{4.58}$$

for any $t<T^*$ and $\mathcal{H}(t)$ given by (4.52).

Proof. Since $\operatorname{div} u=0$, similar to the derivation (4.12), for $p>4$, one has

$$\frac1p\frac{d}{dt}\|\dot\Delta_j\theta(t)\|^p_{L^p} + \int_{\mathbb{R}^2}\big(|D|\dot\Delta_j\theta\big)|\dot\Delta_j|^{p-2}\dot\Delta_j\theta\,dx \leq \|\dot\Delta_j\theta\|^{p-1}_{L^p}\|[\dot\Delta_j;u\cdot\nabla]\theta\|_{L^p},$$

from which and the generalized Bernstein inequality (see [20, 31] for instance)

$$c2^j\|\dot\Delta_j\theta\|^p_{L^p} \leq \int_{\mathbb{R}^2}\big(|D|\dot\Delta_j\theta\big)|\dot\Delta_j|^{p-2}\dot\Delta_j\theta\,dx,$$

we infer

$$\|\dot\Delta_j\theta(t)\|_{L^p} \leq \|\dot\Delta_j\theta_0\|_{L^p}\exp\big(-c2^jt\big) + \int_0^t \exp\big(-c2^j(t-t')\big)\|[\dot\Delta_j;u\cdot\nabla]\theta(t')\|_{L^p}\,dt'. \tag{4.59}$$

Using Bony's decomposition, we write

$$[\dot\Delta_j;u\cdot\nabla]\theta = [\dot\Delta_j;T_u\cdot\nabla]\theta + \dot\Delta_jT_{\nabla\theta}u - T_{\dot\Delta_j\nabla\theta}u + \dot\Delta_jR(u,\nabla\theta) - R(u,\dot\Delta_j\nabla\theta),$$

from which, we infer, by a similar derivation of (4.37), that

$$\|[\dot\Delta_j;u\cdot\nabla]\theta(t')\|_{L^p} \leq C2^{-\frac j2}\|\nabla u(t')\|_{L^\infty}\|\theta(t')\|_{\dot B^{1/2}_{p,\infty}}.$$

Resuming the above estimate into (4.59) and using Definition 3.2, we obtain

$$\|\theta\|_{\widetilde L^\infty_t(\dot B^{1/2}_{p,\infty})} + \|\theta\|_{\widetilde L^1_t(\dot B^{3/2}_{p,\infty})} \leq C\Big(\|\theta_0\|_{\dot B^{1/2}_{p,\infty}} + \int_0^t\|\nabla u(t')\|_{L^\infty}\|\theta(t')\|_{\dot B^{1/2}_{p,\infty}}\,dt'\Big).$$

Applying Gronwall's inequality and using (4.52) leads to (4.58). □

Finally by applying Lemma 3.2, we prove the following *a priori* estimate for smooth enough solutions of (1.1):

Corollary 4.3. *Let (θ, u) be a smooth enough solution of the System* (1.1) *on* $[0, T^*[$. *Then under the assumptions of Theorem 1.1, for $t < T^*$, we have*

$$\|u\|_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})} \leq C\Big(\|u_0\|_{H^1} + CE_0\big((1+E_0^2)(1+\mathcal{G}_2^{\frac14}) + \exp\big(\frac{p}{2(p-4)}\mathcal{G}_1\big)\big)\Big) \stackrel{\text{def}}{=} \mathcal{G}_5 \tag{4.60}$$

for E_0 and $\mathcal{G}_1, \mathcal{G}_2$ given by (2.1) *and* (4.39), (4.40) *respectively.*

Proof. In view of (1.1), we get, by applying Lemma 3.2, that

$$\begin{aligned}\|u\|_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})} \leq & C\Big(\|u_0\|_{\dot{B}^{1/2}_{2,\infty}} + \|\theta\|_{L^{\frac43}_t(L^2)} + \|\nabla u\|_{L^2_t(L^4)}\|u\|_{L^\infty_t(L^2)} \\ & + \|\mu(\theta)-1\|_{L^\infty_t(L^\infty)}\|u\|_{\widetilde{L}^2_t(\dot{B}^{3/2}_{2,\infty})} + \|\theta\|^{\frac{p}{p-4}}_{L^\infty_t(\dot{B}^{1/2}_{p,\infty})}\|\nabla u\|_{L^2_t(L^2)}\Big).\end{aligned} \tag{4.61}$$

In view of (1.7), one has

$$\|\theta\|_{L^{\frac43}_t(L^2)} \lesssim E_0.$$

While we deduce from (4.29) and (4.40) that

$$\|\nabla u\|_{L^2_t(L^4)} \lesssim (1+E_0)E_0^{\frac12}\big(1+\mathcal{G}_2^{\frac14}\big).$$

Substituting the above inequalities, (2.1) and (4.40) into (4.61) and taking ε_0 to be sufficiently small in (1.5), we achieve (4.60). □

5. Proof of Theorem 1.1

5.1. **Existence part of Theorem 1.1.** We first prove the propagation of low regularities for the temperature function θ.

Lemma 5.1. *Let (θ, u) be a smooth solution of system* (1.1) *on* $[0, T^*[$. *Then under the assumptions of Theorem 1.1, we have for any $t < T^*$*

$$\|\theta\|_{\widetilde{L}^\infty_t(\dot{H}^{-s_0})} \leq CE_0\big(1 + E_0(1+E_0+\mathcal{G}_1)\big) \stackrel{\text{def}}{=} \mathcal{G}_6 \tag{5.1}$$

for E_0 and $\mathcal{G}_1$ given by (1.8) *and* (4.39) *respectively.*

Proof. By virtue of (4.59), we have

$$\|\dot{\Delta}_j\theta(t)\|_{L^2} \lesssim e^{-ct2^j}\|\dot{\Delta}_j\theta_0\|_{L^2} + \int_0^t e^{-c(t-t')2^j}\|[\dot{\Delta}_j; u\cdot\nabla]\theta(t')\|_{L^2}\,dt', \tag{5.2}$$

from which, we infer

$$\begin{aligned}\|\theta\|_{\widetilde{L}^\infty_t(\dot{H}^{-s_0})} &\lesssim \|\theta_0\|_{\dot{H}^{-s_0}} + \Big(\sum_{j\in\mathbb{Z}} 2^{-2js_0}\|[\dot{\Delta}_j; u\cdot\nabla]\theta\|^2_{L^1_t(L^2)}\Big)^{\frac12} \\ &\lesssim \|\theta_0\|_{\dot{H}^{-s_0}} + \int_0^t \sum_{j\in\mathbb{Z}} 2^{-js_0}\|[\dot{\Delta}_j; u\cdot\nabla]\theta(t')\|_{L^2}\,dt'.\end{aligned}$$

Applying Lemma 3.3 gives rise to

$$\|\theta\|_{\widetilde{L}^\infty_t(\dot{H}^{-s_0})} \lesssim \|\theta_0\|_{\dot{H}^{-s_0}} + \|\theta\|_{L^2_t(\dot{H}^{1-s_0})}\|\nabla u\|_{L^2_t(L^2)}. \tag{5.3}$$

By the same manner, we get

$$\begin{aligned}\|\theta\|_{L^2_t(\dot{H}^{1-s_0})} \lesssim & \|\theta_0\|_{\dot{H}^{1/2-s_0}} + \int_0^t \sum_{j\in\mathbb{Z}} 2^{-j(s_0-1/2)}\|[\dot{\Delta}_j; u\cdot\nabla]\theta(t')\|_{L^2}\,dt' \\ \lesssim & \|\theta_0\|_{\dot{H}^{1/2-s_0}} + \|\theta\|_{L^2_t(\dot{H}^{3/2-s_0})}\|\nabla u\|_{L^2_t(L^2)} \\ \lesssim & \|\theta_0\|_{\dot{H}^{-s_0}\cap L^2} + E_0\|\theta\|_{L^2_t(\dot{H}^{3/2-s_0})},\end{aligned}$$

and

$$\begin{aligned}\|\theta\|_{L^2_t(\dot H^{3/2-s_0})}\lesssim&\|\theta_0\|_{\dot H^{1-s_0}}+\|\theta\|_{L^2_t(\dot H^{2-s_0})}\|\nabla u\|_{L^2_t(L^2)}\\ \lesssim&\|\theta_0\|_{\dot H^{-s_0}\cap L^2}+E_0\|\theta\|_{L^2_t(L^1\cap\dot H^1)}.\end{aligned}$$

On the other hand, note that Inequalities (1.7) and (4.40) ensures that

$$\|\theta\|_{L^2_t(L^2\cap\dot H^1)}\lesssim E_0+\mathcal{G}_1,$$

so that we achieve

$$\|\theta\|_{L^2_t(\dot H^{1-s_0})}\lesssim E_0(1+E_0+\mathcal{G}_1),$$

from which and (5.3), we obtain (5.1). This completes the proof of the proposition. □

Now we are in a position to complete the existence part of Theorem 1.1:

Existence part of Theorem 1.1. The proof of the existence of solutions to a nonlinear partial differential equations basically follows from the following strategy: one begins by solving an appropriate approximate problem and then proves the uniform bounds for such approximate solutions, and the last step consists in proving the convergence of such approximate solutions to a solution of the original system.

According to this strategy, we first mollify the initial data as follows:

$$\theta_0^n\stackrel{\text{def}}{=}(\dot S_n-\dot S_{-n})\theta_0\quad\text{and}\quad u_0^n\stackrel{\text{def}}{=}(\dot S_n-\dot S_{-n})u_0\qquad\text{for}\qquad n\in\mathbb{N}^*.$$

Then we have $(\theta_0^n,u_0)\in B^{s+1}_{2,1}\times B^s_{2,1}$ for any $s>0$. Moreover, by virtue of (1.5), there holds

$$\begin{aligned}\|\mu(\theta_0^n)-1\|_{L^\infty}\leq&\|\mu(\theta_0^n)-\mu(\theta_0)\|_{L^\infty}+\|\mu(\theta_0)-1\|_{L^\infty}\\ \lesssim&\|\theta_0^n-\theta_0\|_{\dot H^{1/2}}+\|\theta_0^n-\theta_0\|_{\dot B^{1/2}_{p,\infty}}+\|\mu(\theta_0)-1\|_{L^\infty}\qquad\text{for}\qquad 4<p\\ \lesssim&\varepsilon_0.\end{aligned}\tag{5.4}$$

In view of Theorem A.1 in the Appendix, we deduce that for any $s\in[1/2,2]$, the System (1.1) with the initial data (θ_0^n,u_0^n) admits a unique local in time solution $(\theta^n,u^n,\nabla\Pi^n)$ on $[0,T_*^n[$ which verify

$$\begin{aligned}\theta^n\in\mathcal{C}([0,T^n[;B^{1+s}_{2,1})\cap L^1_{T^n}(B^{2+s}_{2,1}),\ u^n\in\mathcal{C}([0,T^n[;\,B^s_{2,1})\cap L^1_{T^n}(B^{2+s}_{2,1})\\ \text{and}\quad\nabla\Pi^n\in L^1_{T^n}(B^s_{2,1})\quad\text{for any}\quad T^n<T^n_*.\end{aligned}$$

Furthermore, it follows from Proposition 4.1 that for any $t<T_*^n$

$$\|\theta^n(t)\|_{L^2}\leq CE_0\langle t\rangle^{-s_0}\quad\text{and}\quad\|u^n\|_{L^\infty_t(L^2)}+\|\nabla u^n\|_{L^2_t(L^2)}\leq CE_0\tag{5.5}$$

for E_0 given by (2.1).

And Proposition 4.2 and Lemma 5.1 ensure that for $p\in[4,p^*]$ and for any $t<T_*^n$

$$\begin{aligned}&\|\theta^n\|^2_{\widetilde L^\infty_t(\dot H^{1/2})}+\ln\bigl(1+\|\theta^n\|_{L^\infty_t(\dot B^{1/2}_{p,\infty})}\bigr)+\|\theta\|^2_{L^2_t(\dot H^1)}\leq\mathcal{G}_1,\quad\|\nabla u^n\|_{L^2_t(L^p)}\leq\mathcal{G}_3\\ &\|\nabla u^n\|^2_{L^\infty_t(L^2)}+\|\partial_t u^n\|^2_{L^2_t(L^2)}\leq\mathcal{G}_2\quad\text{and}\quad\|\theta\|_{\widetilde L^\infty_t(\dot H^{-s_0})}\leq\mathcal{G}_6\end{aligned}\tag{5.6}$$

for $\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3$ given by (4.39) and (4.40) respectively and for $\mathcal{G}_6$ given by (5.1).

Proposition 4.3, Corollaries 4.2 and 4.3 imply that for any $t<T_*^n$

$$\begin{aligned}\|u^n\|_{L^1_t(\dot B^1_{\infty,1})}\leq&\mathcal{H}(t),\\ \|\theta^n\|_{\widetilde L^\infty_t(\dot B^{1/2}_{p,\infty})}+\|\theta^n\|_{\widetilde L^1_t(\dot B^{3/2}_{p,\infty})}\leq&\|\theta_0\|_{\dot B^{1/2}_{p,\infty}}\exp\bigl(C\mathcal{H}(t)\bigr),\\ \|u\|_{\widetilde L^2_t(\dot B^{3/2}_{2,\infty})}\leq&\mathcal{G}_5\end{aligned}\tag{5.7}$$

for $\mathcal{H}(t)$ given by (4.52) and $\mathcal{G}_5$ given by (4.60). Therefore according to Theorem A.1 in the Appendix, $T_*^n=\infty$.

With the uniform estimate (5.5) to (5.7) in hand, we can prove that there is a a subsequence of $\big\{(\theta^n, u^n, \nabla\Pi^n)\big\}_{n\in\mathbb{N}}$, which converges to a solution $(\theta, u, \nabla\Pi)$ of (1.1) by using a standard compactness argument of Lions-Aubin's Lemma. Moreover, this solution satisfies (1.6) and (1.7). Since this argument is rather standard, we shall not present the details here. For instance, one may check similar argument from page 582 to page 583 of [2] for details. $\square$

5.2. **Uniqueness part of Theorem 1.1.** It remains to prove the uniqueness part of Theorem 1.1, which we present now.

Proof of the uniqueness part of Theorem 1.1. Let $(\theta_i, u_i, \nabla\Pi_i)$, for $i = 1, 2$, be two solutions of (1.1) which satisfies (1.6). We denote

$$(\delta\theta, \delta u, \nabla\delta\Pi) \stackrel{\text{def}}{=} (\theta_2 - \theta_1, u_2 - u_1, \nabla\Pi_2 - \nabla\Pi_1).$$

Then by virtue of (1.1), the system for $(\delta\theta, \delta u, \nabla\delta\Pi)$ reads

$$\begin{cases} \partial_t\delta\theta + u_2\cdot\nabla\delta\theta + |D|\delta\theta = -\delta u\cdot\nabla\theta_1 \\ \partial_t\delta u + (u_2\cdot\nabla)\delta u - 2\mathrm{div}\big(\mu(\theta_2)d(\delta u)\big) + \nabla\delta\Pi = \delta\theta e_2 + \delta F, \\ \mathrm{div}\, u = 0, \\ (\delta\theta, \delta u)|_{t=0} = (0,0), \end{cases} \tag{5.8}$$

where δF is determined by

$$\delta F = -(\delta u\cdot\nabla)u_1 + 2\mathrm{div}\big((\mu(\theta_2) - \mu(\theta_1))d(u_1)\big).$$

Note that since $p > 4$, one has

$$\|\nabla\theta_1\|_{L^1_t(L^\infty)} \lesssim \|\theta\|^{\frac12-\frac2p}_{\widetilde{L}^1_t(B^{1/2}_{p,\infty})}\|\theta\|^{\frac12+\frac2p}_{\widetilde{L}^1_t(B^{3/2}_{p,\infty})},$$

then we get, by using energy estimate to the $\delta\theta$ equation of (5.8), that

$$\begin{aligned} \|\delta\theta\|^2_{L^\infty_t(L^2)} + \|\delta\theta\|^2_{L^2_t(\dot H^{1/2})} &\le -\int_0^t \big(\delta u\cdot\nabla\theta_1 \mid \delta u\big)\,dt' \\ &\le \int_0^t \|\nabla\theta_1(t')\|_{L^\infty}\|\delta u(t')\|_{L^2}\|\delta\theta(t')\|_{L^2}\,dt'. \end{aligned} \tag{5.9}$$

While it follows from Proposition 3.1 that

$$\|\delta\theta\|^2_{L^\infty_t(\dot B^0_{\frac{2p}{p-2},\infty})} \lesssim \|\delta u\cdot\nabla\theta_1\|^2_{L^2_t(\dot B^{-1/2}_{\frac{2p}{p-2},\infty})}\exp\Big(C\|\nabla u_2\|_{L^2_t(L^4)}\Big).$$

Yet applying Bony's decomposition (3.5) gives

$$\delta u\cdot\nabla\theta_1 = T_{\delta u}\nabla\theta_1 + T_{\nabla\theta_1} + R(\delta u, \nabla\theta_1)$$

from which, and para-product estimates in [7], we infer

$$\|\delta u(t)\cdot\nabla\theta_1(t)\|_{\dot B^{-1/2}_{\frac{2p}{p-2},\infty}} \lesssim \|\delta u(t)\|_{L^{\frac{2p}{p-4}}}\|\theta_1(t)\|_{B^{1/2}_{p,\infty}}.$$

Note that in $\mathbb{R}^2$, one has

$$\|\delta u(t)\|_{L^{\frac{2p}{p-4}}} \le C\|\delta u(t)\|^{\frac{p-4}{p}}_{L^2}\|\nabla\delta u(t)\|^{\frac4p}_{L^2},$$

so that applying Young's inequality leads to

$$\|\delta\theta\|^2_{L^\infty_t(\dot B^0_{\frac{2p}{p-2},\infty})} \le C\exp\Big(C\|\nabla u_2\|_{L^2_t(L^4)}\Big)\int_0^t \|\theta_1\|^{\frac{2p}{p-4}}_{\dot B^{1/2}_{p,\infty}}\|\delta u\|^2_{L^2}\,dt' + \frac18\|\nabla\delta u\|^2_{L^2_t(L^2)}. \tag{5.10}$$

Whereas by taking L^2 inner product δu with the δu equation of (5.8), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\delta u|^2\,dx+2\int_{\mathbb{R}^2}\mu(\theta_2)d(\delta u):d(\delta u)\,dx=\int_{\mathbb{R}^2}(\delta\theta e_2+\delta F)\mid\delta u\,dx,$$

which together with

$$\begin{aligned}\Big|\int_{\mathbb{R}^2}\delta u\cdot\nabla u_1\mid\delta u(t)\,dx\Big| &\leq \|\delta u\cdot\nabla u_1(t)\|_{\mathscr{H}^1}\|\delta u(t)\|_{BMO}\\ &\leq C\|\nabla u_1(t)\|_{L^2}\|\delta u(t)\|_{L^2}\|\nabla\delta u(t)\|_{L^2},\end{aligned}$$

leads to

$$\begin{aligned}\|\delta u\|_{L^\infty_t(L^2)}^2+\|\nabla\delta u\|_{L^2_t(L^2)}^2\leq &C\Big(\int_0^t\|\nabla u_1(t')\|_{L^2}^2\|\delta u(t')\|_{L^2}^2\,dt'\\ &+\int_0^t\|\delta\theta\|_{L^{\frac{2p}{p-2}}}\|\nabla u_1\|_{L^p}\|\nabla\delta u\|_{L^2}\,dt'+\int_0^t\|\delta\theta\|_{L^2}\|\delta u\|_{L^2}\,dt'\Big).\end{aligned}\tag{5.11}$$

Notice that for any positive integer N and $p>4$, we write

$$\|\delta\theta(t)\|_{L^{\frac{2p}{p-2}}}\leq\|\delta\theta(t)\|_{\dot B^0_{\frac{2p}{p-2},2}}\leq\|\delta\theta(t)\|_{L^2}+\sqrt{N}\|\delta\theta(t)\|_{\dot B^0_{\frac{2p}{p-2},\infty}}+2^{-N(\frac{1}{2}-\frac{2}{p})}\|\delta\theta(t)\|_{\dot H^{1/2}}.$$

Taking N in the above inequality so that

$$2^{N(\frac{1}{2}-\frac{2}{p})}\sim\|\delta\theta(t)\|_{\dot H^{1/2}}\Big/\|\delta\theta(t)\|_{\dot B^0_{\frac{2p}{p-2},\infty}}$$

we obtain

$$\|\delta\theta(t)\|_{L^{\frac{2p}{p-2}}}\leq C\Big(\|\delta\theta(t)\|_{L^2}+\|\delta\theta(t)\|_{\dot B^0_{\frac{2p}{p-2},\infty}}\ln^{\frac{1}{2}}\big(e+\sum_{i=1}^2\|\theta_i\|_{\dot H^{1/2}}\Big/\|\delta\theta(t)\|_{\dot B^0_{\frac{2p}{p-2},\infty}}\big)\Big).$$

Substituting the above inequality into (5.11) gives rise to

$$\begin{aligned}&\|\delta u\|_{L^\infty_t(L^2)}^2+\|\nabla\delta u\|_{L^2_t(L^2)}^2\\ &\quad\leq C\int_0^t\big(1+\|\nabla u_1(t')\|_{L^2}^2+\|\nabla u_1(t')\|_{L^p}^2\big)\big(\|\delta\theta(t')\|_{L^2}^2+\|\delta u(t')\|_{L^2}^2\big)\,dt'\\ &\quad\quad+C\int_0^t\|\nabla u_1(t')\|_{L^p}^2\|\delta\theta(t')\|_{\dot B^0_{\frac{2p}{p-2},\infty}}^2\ln\big(e+\sum_{i=1}^2\|\theta_i(t')\|_{\dot H^{1/2}}\Big/\|\delta\theta(t')\|_{\dot B^0_{\frac{2p}{p-2},\infty}}\big)\,dt'\\ &\quad\quad+\frac{1}{8}\|\nabla\delta u\|_{L^2_t(L^2)}^2.\end{aligned}\tag{5.12}$$

Let us denote

$$Y(t)\stackrel{\text{def}}{=}\|\delta\theta\|_{L^\infty_t(L^2)}^2+\|\delta\theta\|_{L^\infty_t(\dot B^0_{\frac{2p}{p-2},\infty})}^2+\|\delta u\|_{L^\infty_t(L^2)}^2.\tag{5.13}$$

Then for $\beta(t)$ given by

$$\beta(t)\stackrel{\text{def}}{=}\sum_{i=1}^2\|\theta_i(t)\|_{\dot H^{1/2}},$$

by summing up (5.9), (5.10) and (5.12), we obtain

$$\begin{aligned}Y(t)\leq C\exp\Big(C\|\nabla u_2\|_{L^2_t(L^4)}\Big)&\int_0^t\big(1+\|\nabla\theta_1(t')\|_{L^\infty}+\|\nabla u_1(t')\|_{L^2}^2\\ &+\|\nabla u_1(t')\|_{L^p}^2+\|\theta_1(t')\|_{\dot B^{1/2}_{p,\infty}}^{\frac{2p}{p-4}}\big)Y(t')\ln\Big(e+\beta(t')\Big/Y(t')\Big)\,dt',\end{aligned}$$

from which and Osgood Lemma, we infer

$$Y(t) = 0.$$

This completes the uniqueness part of Theorem 1.1. $\square$

5.3. **The proof of Corollary 1.1.** Indeed, let $h(\theta) \stackrel{\text{def}}{=} \mu(\theta) - 1$. Then under the assumption (1.10), we deduce from from the θ equation of (1.1) and Lemma 4.1 that

$$\|\mu(\theta(t)) - 1\|_{L^\infty} = \|h(\theta(t))\|_{L^\infty} \leq \|h(\theta_0)\|_{L^\infty} = \|\mu(\theta_0) - 1\|_{L^\infty},$$

which implies (1.5). Thus according to Theorem 1.1, we complete the proof of Corollary 1.1.

Appendix A. The blow-up criteria for smooth solutions of (1.1)

The goal of this section is to prove the following local well-posedness result of (1.1):

Theorem A.1. *Let $s \geq 1/2$, $\theta_0 \in B^{1+s}_{2,1}$ and $u_0 \in B^s_{2,1}$. Then there exists s positive time T^* so that* (1.1) *has a unique solution (θ, u) on $[0, T^*[$ so that for any $T < T^*$, one has*

$$\theta \in \mathcal{C}([0,T]; B^{1+s}_{2,1}) \cap L^1_T(B^{2+s}_{2,1}) \quad \text{and} \quad u \in \mathcal{C}([0,T]; B^s_{2,1}) \cap L^1_T(B^{2+s}_{2,1}). \tag{A.1}$$

Furthermore, if T^* is the maximal time of existence to this solution, and $T^* < \infty$, then one has

$$\lim_{t \to T^*} \|\nabla u\|_{L^1_t(L^\infty)} = \infty.$$

Proof. It follows by a standard argument (see for instance the proof of Theorems 1.1 and 1.2 in [2]) that (1.1) has a unique local solution (θ, u) on $[0, T^*[$ so that there holds (A.1). It remains to prove that if

$$\lim_{t \to T^*} \|\nabla u\|_{L^1_t(L^\infty)} < \infty, \tag{A.2}$$

one has

$$\lim_{t \to T^*} \Big(\|\theta(t)\|_{B^{1+s}_{2,1}} + \|u(t)\|_{B^s_{2,1}} \Big) < \infty. \tag{A.3}$$

As a matter of fact, we first deduce from a standard energy estimate to (1.1) that

$$\frac{1}{2} \frac{d}{dt} \Big(\|\theta(t)\|^2_{L^2} + \|u(t)\|^2_{L^2} \Big) + \|\theta\|^2_{\dot{H}^{1/2}} + \int_{\mathbb{R}^2} \mu(\theta) d : d \, dx = \int_{\mathbb{R}^2} \theta u_2 \, dx,$$

which together with (1.4) implies

$$\frac{1}{2} \Big(\|\theta(t)\|^2_{L^2} + \|u(t)\|^2_{L^2} \Big) + \|\theta\|^2_{L^2_t(\dot{H}^{1/2})} + \|\nabla u\|^2_{L^2_t(L^2)} \leq \frac{1}{2} \Big(\|\theta_0\|^2_{L^2} + \|u_0\|^2_{L^2} \Big) e^t. \tag{A.4}$$

As a convention in the remaining of this section, we shall always assume that $t < T^*$.

step 1. The estimate of $\|\nabla \theta\|_{L^2_t(L^\infty)}$.

For any $p \in]1, \infty[$, we first deduce, by a similar derivation of (3.18), that

$$\|\dot{\Delta}_j \theta\|_{L^\infty_t(L^p)} + c 2^j \|\dot{\Delta}_j \theta\|_{L^1_t(L^p)} \leq \|\dot{\Delta}_j \theta_0\|_{L^p} + \int_0^t \|[\dot{\Delta}_j; u \cdot \nabla]\theta(t')\|_{L^p} \, dt'. \tag{A.5}$$

Using Bony's decomposition (3.5), we write

$$[\dot{\Delta}_j; u \cdot \nabla]\theta = [\dot{\Delta}_j; T_u \nabla]\theta + \dot{\Delta}_j T'_{\nabla \theta} u - T'_{\nabla \dot{\Delta}_j \theta} u.$$

It follows from the classical commutator's estimate that

$$\begin{aligned} \|[\dot{\Delta}_j; T_u \nabla]\theta(t)\|_{L^p} &\lesssim \sum_{|j'-j| \leq 4} \|S_{j'-1} \nabla u(t)\|_{L^\infty} \|\dot{\Delta}_j \theta(t)\|_{L^p} \\ &\lesssim d_j(t) 2^{-j s_1} \|\nabla u(t)\|_{L^\infty} \|\theta(t)\|_{\dot{B}^{s_1}_{p,1}}. \end{aligned}$$

While for $s_1 \in]0,1[$, applying Lemma 3.1 gives

$$\begin{aligned}\|T'_{\nabla\theta}u(t)\|_{L^p} &\lesssim \sum_{j'\geq j-N_0} \|S_{j'+2}\nabla\theta(t)\|_{L^p}\|\dot\Delta_{j'}u(t)\|_{L^\infty}\\ &\lesssim \sum_{j'\geq j-N_0} d_{j'}(t)2^{j'(1-s_1)}\|\theta(t)\|_{\dot B^{s_1}_{p,1}}2^{-j'}\|\nabla u(t)\|_{L^\infty}\\ &\lesssim d_j(t)2^{-js_1}\|\nabla u(t)\|_{L^\infty}\|\theta(t)\|_{\dot B^{s_1}_{p,1}}.\end{aligned}$$

The same estimate holds for $T'_{\nabla\dot\Delta_j\theta}u$. Whence we obtain

$$\|[\dot\Delta_j; u\cdot\nabla]\theta(t)\|_{L^p} \lesssim d_j(t)2^{-js_1}\|\nabla u(t)\|_{L^\infty}\|\theta(t)\|_{\dot B^{s_1}_{p,1}},$$

from which and (A.5), we infer

$$\|\theta\|_{\widetilde L^\infty_t(\dot B^{s_1}_{p,1})} + \|\theta\|_{L^1_t(\dot B^{1+s_1}_{p,1})} \leq C\Big(\|\theta_0\|_{\dot B^{s_1}_{p,1}} + \int_0^t \|\nabla u(t')\|_{L^\infty}\|\theta(t')\|_{\dot B^{s_1}_{p,1}}\,dt'\Big) \quad \forall\, s_1 \in]0,1[. \tag{A.6}$$

In particular, for any $p \in]4,\infty[$, by taking $s_1 = 1/2+2/p$ in the above inequality and then applying Gronwall's inequality, we achieve

$$\|\theta\|_{\widetilde L^\infty_t(\dot B^{1/2+2/p}_{p,1})} + \|\theta\|_{L^1_t(\dot B^{3/2+2/p}_{p,1})} \leq C\|\theta_0\|_{\dot B^{1/2+2/p}_{p,1}} \exp\Big(C\|\nabla u\|_{L^1_t(L^\infty)}\Big), \tag{A.7}$$

and hence due to $s \geq 1/2$, we get, by applying Lemma 3.1, that

$$\begin{aligned}\|\nabla\theta\|_{L^2_t(L^\infty)} \leq \|\theta\|_{\widetilde L^2_t(\dot B^{1+2/p}_{p,1})} &\leq C\|\theta\|^{\frac12}_{\widetilde L^\infty_t(\dot B^{1/2+2/p}_{p,1})}\|\theta\|^{\frac12}_{L^1_t(\dot B^{3/2+2/p}_{p,1})}\\ &\leq C\|\theta_0\|_{B^{1+s}_{2,1}}\exp\Big(C\|\nabla u\|_{L^1_t(L^\infty)}\Big).\end{aligned} \tag{A.8}$$

step 2. The estimate of $\|\theta\|_{\widetilde L^\infty_t(\dot B^s_{2,1})}$ and $\|\theta\|_{\widetilde L^\infty_t(\dot B^{1+s}_{2,1})}$.

We first get, by a similar derivation of (A.6), that for any $s_2 > 0$,

$$\|\theta\|_{\widetilde L^\infty_t(\dot B^{s_2}_{2,1})} + \|\theta\|_{L^1_t(\dot B^{1+s_2}_{2,1})} \leq C\Big(\|\theta_0\|_{\dot B^{s_2}_{2,1}} + \int_0^t \big(\|\nabla u\|_{L^\infty}\|\theta\|_{\dot B^{s_2}_{2,1}} + \|\nabla\theta\|_{L^\infty}\|u\|_{\dot B^{s_2}_{2,1}}\big)(t')\,dt'\Big), \tag{A.9}$$

Taking $s_2 = s$ and $s_2 = 1+s$ in (A.9) and then summing up the resulting inequalities results in

$$\begin{aligned}&\|\theta\|_{\widetilde L^\infty_t(\dot B^s_{2,1})} + \|\theta\|_{\widetilde L^\infty_t(\dot B^{1+s}_{2,1})} + \|\theta\|_{L^1_t(\dot B^{1+s}_{2,1})} + \|\theta\|_{L^1_t(\dot B^{2+s}_{2,1})}\\ &\leq C\Big(\|\theta_0\|_{\dot B^s_{2,1}} + \|\theta_0\|_{\dot B^{1+s}_{2,1}} + \int_0^t\Big(\|\nabla u\|_{L^\infty}\big(\|\theta\|_{\dot B^s_{2,1}} + \|\theta\|_{\dot B^{1+s}_{2,1}}\big)\\ &\qquad + \|\nabla\theta\|_{L^\infty}(1+\|\nabla\theta\|_{L^\infty})\|u\|_{\dot B^s_{2,1}}\Big)(t')\,dt'\Big) + \frac14\|u\|_{L^1_t(\dot B^{2+s}_{2,1})},\end{aligned} \tag{A.10}$$

where we used the fact that

$$\|u(t)\|_{\dot B^{1+s}_{2,1}} \lesssim \|u(t)\|^{\frac12}_{\dot B^s_{2,1}}\|u(t)\|^{\frac12}_{\dot B^{2+s}_{2,1}}. \tag{A.11}$$

step 3. The estimate of $\|u\|_{\widetilde L^\infty_t(\dot B^s_{2,1})}$.

We first deduce from (4.35) and Definition 3.2 that

$$\begin{aligned}\|u\|_{\widetilde L^\infty_t(\dot B^s_{2,1})} + \|u\|_{L^1_t(\dot B^{2+s}_{2,1})} \lesssim& \|u_0\|_{\dot B^s_{2,1}} + \sum_{j\in\mathbb Z} 2^{js}\|[\dot\Delta_j; u\cdot\nabla]u\|_{L^1_t(L^2)}\\ &+ \sum_{j\in\mathbb Z} 2^{j(1+s)}\|[\dot\Delta_j; \mu(\theta)\cdot\nabla]u\|_{L^1_t(L^2)} + \sum_{j\in\mathbb Z} 2^{js}\|\dot\Delta_j\theta\|_{L^1_t(L^2)}.\end{aligned} \tag{A.12}$$

The same proof of (4.37) ensures that

$$\sum_{j\in\mathbb{Z}} 2^{js}\|[\dot{\Delta}_j; u\cdot\nabla]u\|_{L^1_t(L^2)} \lesssim \int_0^t \|\nabla u(t')\|_{L^\infty}\|u(t')\|_{\dot{B}^s_{2,1}}\,dt'.$$

While by using Bony's decomposition (3.5), we write

$$[\dot{\Delta}_j;\mu(\theta)\cdot\nabla]u = [\dot{\Delta}_j; T_{\mu(\theta)}]\nabla u + \dot{\Delta}_j(T'_{\nabla u}(\mu(\theta)-1)) - T'_{\nabla\dot{\Delta}_j u}(\mu(\theta)-1).$$

Applying classical commutator's estimate yields

$$\begin{aligned}\|[\dot{\Delta}_j; T_{\mu(\theta)}]\nabla u(t)\|_{L^2} \lesssim& 2^{-j}\sum_{|j'-j|\leq 4}\|S_{j'-1}\nabla\mu(\theta(t))\|_{L^\infty}\|\dot{\Delta}_j\nabla u(t)\|_{L^2}\\ \lesssim& d_j(t)2^{-j(1+s)}\|\nabla\theta(t)\|_{L^\infty}\|u(t)\|_{\dot{B}^{1+s}_{2,1}}.\end{aligned}$$

Whereas applying Lemma 3.1 leads to

$$\begin{aligned}\|\dot{\Delta}_j(T'_{\nabla u}(\mu(\theta)-1))(t)\|_{L^2} \lesssim& \sum_{j'\geq j-N_0}\|S_{j'+2}\nabla u(t)\|_{L^\infty}\|\dot{\Delta}_{j'}(\mu(\theta(t))-1)\|_{L^2}\\ \lesssim& d_j(t)2^{-j(1+s)}\|\nabla u(t)\|_{L^\infty}\|\theta(t)\|_{\dot{B}^{1+s}_{2,1}}.\end{aligned}$$

The same estimate holds for $T'_{\nabla\dot{\Delta}_j u}(\mu(\theta)-1)$. So that we achieve

$$\sum_{j\in\mathbb{Z}} 2^{j(1+s)}\|[\dot{\Delta}_j;\mu(\theta)\cdot\nabla]u\|_{L^1_t(L^2)} \lesssim \int_0^t \big(\|\nabla\theta\|_{L^\infty}\|u\|_{\dot{B}^{1+s}_{2,1}} + \|\nabla u\|_{L^\infty}\|\theta\|_{\dot{B}^{1+s}_{2,1}}\big)(t')\,dt'.$$

Resuming the above estimates into (A.12) gives rise to

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_t(\dot{B}^s_{2,1})} + \|u\|_{L^1_t(\dot{B}^{2+s}_{2,1})} \lesssim& \|u_0\|_{\dot{B}^s_{2,1}} + \int_0^t \big(\|\nabla u\|_{L^\infty}(\|u\|_{\dot{B}^s_{2,1}}\\ &+\|\theta\|_{\dot{B}^{1+s}_{2,1}}) + \|\nabla\theta\|_{L^\infty}\|u\|_{\dot{B}^{1+s}_{2,1}}\big)(t')\,dt' + \|\theta\|_{L^1_t(\dot{B}^s_{2,1})}.\end{aligned} \tag{A.13}$$

step 4. The closure of the estimate.

Let us denote

$$\eta(t) \stackrel{\text{def}}{=} \|\theta\|_{\widetilde{L}^\infty_t(\dot{B}^s_{2,1})} + \|\theta\|_{\widetilde{L}^\infty_t(\dot{B}^{1+s}_{2,1})} + \|u\|_{\widetilde{L}^\infty_t(\dot{B}^s_{2,1})} + \|\theta\|_{L^1_t(\dot{B}^{1+s}_{2,1})} + \|\theta\|_{L^1_t(\dot{B}^{2+s}_{2,1})} + \|u\|_{L^1_t(\dot{B}^{2+s}_{2,1})}. \tag{A.14}$$

Then by summing up (A.10) and (A.13), and using (A.11), we conclude

$$\begin{aligned}\eta(t) \leq& C\Big(\|\theta_0\|_{\dot{B}^s_{2,1}} + \|\theta_0\|_{\dot{B}^{1+s}_{2,1}} + \|u_0\|_{\dot{B}^s_{2,1}}\\ &+\int_0^t \big(1+\|\nabla u(t')\|_{L^\infty} + \|\nabla\theta(t')\|_{L^\infty}(1+\|\nabla\theta(t')\|_{L^\infty})\big)\eta(t')\,dt'\Big).\end{aligned}$$

Applying Gronwall's inequality and using (A.2), (A.8) and (A.4) concludes the proof of (A.3). This completes the proof of the theorem. □

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