ON THE GLOBAL WELL-POSEDNESS OF 3-D NAVIER-STOKES EQUATIONS WITH VANISHING HORIZONTAL VISCOSITY

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Abstract. *We study in this paper the axisymmetric 3-D Navier-Stokes system where the horizontal viscosity is zero. We prove the existence of a unique global solution to the system with initial data in Lebesgue spaces.*

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1. Introduction

The classical Navier-Stokes equations describe the evolution of a homogeneous incompressible viscous flow in the three-dimensional space. We recall here those equations:

$$(\mathrm{NS})\quad \begin{cases} \partial_t u + (u\cdot\nabla)u - \nu_h(\partial_x^2+\partial_y^2)u - \nu_v\partial_z^2 u = -\nabla p\\ \operatorname{div} u = 0\\ u_{|t=0} = u_0. \end{cases}$$

Here, by ν_h (respectively ν_v) we denote the horizontal, respectively the vertical viscosity, the velocity of the fluid u is a vector field which depends on the time t and the space variable $x\in\mathbb{R}^3$ and finally, ∇p denotes the corresponding gradient of the pressure which can be interpreted as a Lagrange multiplier associated to the incompressibility condition div $u=0$.

In the case where the viscosity coefficients ν_h and ν_v are strictly positive, it is well known by the J. Leray work [14], that the system (NS) admits a global in time solution in the energy L^2. After these results, H. Fujita and T. Kato [10] have proved that (NS) is locally well posed for general initial data in the homogeneous Sobolev spaces $\dot H^{\frac12}$, by using semi-group techniques. Moreover, they proved the existence of a unique global in time solution, when the initial data is small enough compared with the total viscosity of the system $\inf\{\nu_h,\nu_v\}$. Many other results have been proved in more general functional framework which are all invariant by the parabolic scaling of the equation (see for example [4] and [12]).

In the case where $\nu_h>0$ and $\nu_v=0$ the system $(\mathrm{NS_h})$ has been studied for the first time by J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier in [5]. More precisely, the authors have proved in [5] the local in time existence of the solution when the initial data belongs to the anisotropic Sobolev space $H^{0,\frac12+}$, with $H^{0,s}=\big\{u\in L^2 \mid (\int_{\mathbb{R}^2}\|u(x,y,\cdot)\|^2_{H^s(\mathbb{R})}dxdy)^{\frac12}<\infty\big\}$. The global well-posedness was proved for initial data which are small enough compared with horizontal viscosity ν_h. However, the uniqueness of the solution was proved for more regular initial data, belonging to the space $H^{0,\frac32+}$. The uniqueness in the general case where the initial data verify $u_0\in H^{0,\frac12+}$ was proved later by D. Iftimie [11]. The critical case $s=\frac12$ was studied by M. Paicu [15], who proved that the system $(\mathrm{NS_h})$ is locally well posed in the anisotropic Besov space $\mathscr{B}^{0,\frac12}=\big\{u\in\mathcal{S}' \mid \sum_{q\in\mathbb{Z}}(\int_{2^{q-1}\le|z|\le 2^q}|z|\|\mathcal{F}u(\cdot,\cdot,z)\|^2_{L^2(\mathbb{R}^2)}dz)^{\frac12}<\infty\big\}$, the global existence of the solution was proved for small initial data compared with ν_h. Recently, J.-Y. Chemin and P. Zhang [6] have obtained a similar result by working in an anisotropic Besov space with negative regularity indexes in the horizontal variable. This result allows to prove the global existence of the solution for horizontal Navier-Stokes equations with highly oscillating initial data in the horizontal variables.

In this paper, we study the opposite situation, the case of a vanishing horizontal viscosity and a strictly positive vertical viscosity, namely $\nu_h = 0$ and $\nu_v > 0$. Our main goal in this paper is to obtain the global existence of the solution for very rough initial data. In all of what follows, we suppose for simplicity that the vertical viscosity is constant $\nu_v = 1$, as we are not interested in the dependence of any quantities in the vertical viscosity. In this case, the system becomes:

$$(\mathrm{NS_v})\quad \begin{cases} \partial_t u + (u\cdot\nabla)u - \partial_z^2 u = -\nabla p \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0. \end{cases}$$

We recall that the main idea in the case where $\nu_h > 0$ and $\nu_v = 0$, in order to control the vertical derivative was to use the incompressibility condition, namely $\partial_x u^1 + \partial_y u^2 + \partial_z u^3 = 0$, which allows to obtain a regularizing effect for the vertical component u^3 by using the horizontal viscosity. Contrarily to this situation, our case is more difficult to study because of the lack of regularity in two horizontal variables. By using a regularizing effect only in the vertical direction seems very difficult to recover any regularization in all variables in the general case. This is the main reason for which we restrict ourself to study a particular case, more precisely, we consider only axisymmetric flows. Indeed, for axisymmetric solutions, we have $\operatorname{div} u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0$. Before to go further in the details, it is convenient to precise what exactly we mean by axisymmetric initial data.

Definition 1.1. *We said that the vector field u is axisymmetric ("without swirl"), if and only if, we can write*

$$u = u^r(r,z)e_r + u^z(r,z)e_z$$

where $\big(e_r, e_\theta, e_z\big)$ *is the cylindrical base.*
A scalar function is called axisymmetric if this function has no dependencies on the angular variable θ.

To prove that the solution associated with any initial data u_0 axisymmetric, is axisymmetric, it just uses a method to X. Saint Raymond [17]. The classical Navier-Stokes system (in the case $\nu_h = \nu_v > 0$) has already been studied by many authors, the first results was obtained by M. Ukhovskii and V. Youdovitch [19] and also by O. A. Ladyzhenskaya [13].

In this case, the vorticity of u is defined by $\omega := \nabla \times u$, and admits in a cylindrical frame only one component in the direction of e_θ:

$$\omega = \omega^\theta e_\theta \quad \text{with} \quad \omega^\theta = \partial_z u^r - \partial_r u^z$$

and this vorticity verifies the following equation:

$$\partial_t \omega + (u^r\partial_r + u^z\partial_z)\omega - \frac{u^r}{r}\omega - \partial_z^2\omega = 0,$$

and consequently, ω/r verifies the transport-diffusion equation:

$$\partial_t \frac{\omega^\theta}{r} + (u\cdot\nabla)\frac{\omega^\theta}{r} - \partial_z^2 \frac{\omega^\theta}{r} = 0.$$

Then, it is possible to prove by energy methods that, for all $p \in [1,\infty]$ (respectively $p \in]1,2]$) that the L^p norm of ω/r (resp. $r^{-1}\partial_z\omega$) (respectively the $L^2_t(L^p)$ norm) is controlled by the norm of the initial data ω_0/r. Using the Biot-Savart law, we can prove that (see Proposition 3.1)

$$\Big|\frac{u^r}{r}\Big| \lesssim \frac{1}{|\cdot|} \star |r^{-1}\partial_z\omega|.$$

In this way, the incompressibility condition allows us to control $\partial_r u^r$ by using that $\partial_r u^r = -\frac{u^r}{r} - \partial_z u^z$. In the following we use the notion of Lorentz space which is defined in the next section. Our main result is given bellow:

Theorem 1.1. *Let ω_0 be an axisymmetric function in $L^{\frac{3}{2},1}(\mathbb{R}^3)$ such that $\frac{\omega_0}{r} \in L^{\frac{3}{2},1}(\mathbb{R}^3)$. Let u_0 a axisymmetric solenoidal vector-field with vorticity $\omega_0 e_\theta$ which is given by the Biot-Savart law:*

$$u_0(X) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(X-Y)\times\omega_0(Y)}{|X-Y|^3}\,dY.$$

Then, the system $(\mathrm{NS_v})$ *has a global in time solution u such that the vorticity ω satisfies*

$$\omega \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2},1}(\mathbb{R}^3)\big), \qquad \partial_z\omega \in L^2_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2},1}(\mathbb{R}^3)\big)$$
$$\frac{\omega}{r} \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2},1}(\mathbb{R}^3)\big), \qquad \partial_z\frac{\omega}{r} \in L^2_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2},1}(\mathbb{R}^3)\big).$$

Moreover, for all $t \geq 0$, we have

$$\|\omega(t)\|_{L^{\frac{3}{2},1}} + \|\partial_z\omega\|_{L^2_t(L^{\frac{3}{2},1})} \leq C\|\omega_0\|_{L^{\frac{3}{2},1}} \exp\big(Ct^{\frac{1}{2}}\|r^{-1}\omega_0\|_{L^{\frac{3}{2},1}}\big)$$

and

$$\|r^{-1}\omega(t)\|_{L^{\frac{3}{2},1}} + \|r^{-1}\partial_z\omega\|_{L^2_t(L^{\frac{3}{2},1})} \leq C\|r^{-1}\omega_0\|_{L^{\frac{3}{2},1}}.$$

Furthermore, if $\partial_r\omega_0 \in L^{\frac{3}{2}}$ and $\omega_0 \in L^{3,1}$, then

$$\partial_r\omega \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2}}(\mathbb{R}^3)\big),\ \ \partial_z\partial_r\omega \in L^2_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2}}(\mathbb{R}^3)\big)$$

and the solution is unique.

Remark 1.1. *We recall that R. Danchin* [8] *has proved that the axisymmetric Euler system is globally well posed for initial data with Youdovitch type regularity. More precisely, he proved that the Euler system is globally well posed when the initial vorticity verifies $\omega_0 \in L^{3,1}\cap L^\infty$ and $\omega_0/r \in L^{3,1}$. Recently H. Abidi and al.* [2] *have proved that the axisymmetric Euler system is globally well posed in critical spaces for the initial velocity, more precisely when $u_0 \in B^{\frac{3}{p}+1}_{p,1}$ for $p \in [1,\infty]$ and $\omega_0/r \in L^{3,1}$.*

Remark 1.2. *We note also that H. Abidi obtained previously similar results in* [1]. *Indeed, in this paper, the author proved that the classical axisymmetric Navier-Stokes system (i.e., $\nu_h = \nu_v > 0$) is globally well posed when the initial velocity verifies $u_0 \in W^{2,p}(\mathbb{R}^3)$ for $1 < p < 2$.*

Concerning the existence of the solution we can prove a better result, for even less regular initial data. However, the uniqueness of the solution seems to be more difficult to prove for this weak regularity. Our second result is the following.

In all of what follows, we always make the convention that: for any $\alpha > 0$, α_+ means any constant greater than α.

Theorem 1.2. *Let ω_0 be an axisymmetric function in $L^{\frac{6}{5}} \cap L^{\frac{6}{5}+,1}(\mathbb{R}^3)$ such that $\frac{\omega_0}{r} \in L^{\frac{6}{5}} \cap L^{\frac{6}{5}+,1}(\mathbb{R}^3)$. Let u_0 a axisymmetric solenoidal vector-field with vorticity $\omega_0 e_\theta$ given by Biot-Savart law. Then, the system* $(\mathrm{NS_v})$ *has a global in time solution u such that the vorticity ω satisfies*

$$\big(\omega, \frac{\omega}{r}\big) \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^{\frac{6}{5}} \cap L^{\frac{6}{5}+,1}(\mathbb{R}^3)\big), \qquad \big(\partial_z\omega, \partial_z\frac{\omega}{r}\big) \in L^2_{loc}\big(\mathbb{R}_+;\, L^{\frac{6}{5}} \cap L^{\frac{6}{5}+,1}(\mathbb{R}^3)\big).$$

2. Notations and Preliminaries

We say that $A \lesssim B$ if there exists a positive constant C such that $A \leq CB$. By C we denote a general constant which can change to any line. Let X a Banach space and $p \in [1, \infty]$, we denote by $L^p(0,T;\, X)$ the set of all functions f measurables on $(0,T)$ valued in X, such that $t \longmapsto \|f(t)\|_X$ belongs to $L^p(0,T)$. We denote by $C([0,T);\, X)$ the space of continuous functions from $[0,T)$ valued in X, $C_b([0,T);\, X) \stackrel{\text{déf}}{=} C([0,T);\, X) \cap L^\infty(0,T;\, X)$. Finally we denote by p' the conjugate exponent of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Before to introduce the definition of the Lorentz space, we begin by recalling the rearrangement of a function. For a measurable function f we define its non-increasing rearrangement by $f^* : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f^*(\lambda) := \inf \Big\{ s \geq 0;\ \big|\{x/\,|f(x)| > s\}\big| \leq \lambda \Big\}.$$

Definition 2.1. *(Lorentz spaces) Let f a mesurable function and $1 \leq p, q \leq \infty$. Then f belongs to the Lorentz space $L^{p,q}$ if*

$$\|f\|_{L^{p,q}} \stackrel{def}{=} \begin{cases} \Big(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t} \Big)^{\frac{1}{q}} < \infty & si\ q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) < \infty & si\ q = \infty. \end{cases}$$

Alternatively, we can define the Lorentz spaces by the real interpolation, as the interpolation between the Lebesgue space :

$$L^{p,q} := (L^{p_0}, L^{p_1})_{(\theta,q)},$$

with $1 \leq p_0 < p < p_1 \leq \infty$, $0 < \theta < 1$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $1 \leq q \leq \infty$, also $f \in L^{p,q}$ if the following quantity

$$\|f\|_{L^{p,q}} := \Big(\int_0^\infty \big(t^{-\theta} K(t,f) \big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$

is finite with

$$K(f,t) := \inf_{f = f_0 + f_1} \big\{ \|f_0\|_{L^{p_0}} + t \|f_1\|_{L^{p_1}} \ \big|\ f_0 \in L^{p_0},\ f_1 \in L^{p_1} \big\}.$$

The Lorentz spaces verify the following properties (see [16] for more details) :

Proposition 2.1. *Let $f \in L^{p_1,q_1}$, $g \in L^{p_2,q_2}$ and $1 \leq p, q, p_j, q_j \leq \infty$, for $1 \leq j \leq 2$.*

- *If $1 < p < \infty$ and $1 \leq q \leq \infty$, then*
$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p,q}} \|g\|_{L^\infty}.$$
- *If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*
$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$
- *If $1 < p < \infty$ and $1 \leq q \leq \infty$, then*
$$\|f * g\|_{L^{p,q}} \lesssim \|f\|_{L^{p,q}} \|g\|_{L^1}.$$
- *If $1 < p,\, p_1,\, p_2 < \infty$, $\frac{1}{p} + 1 = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*
$$\|f * g\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}},$$
for $p = \infty$, and $\frac{1}{q_1} + \frac{1}{q_2} = 1$, then
$$\|f * g\|_{L^\infty} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$
- *For $1 \leq p \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, we have*
$$L^{p,q_1} \hookrightarrow L^{p,q_2} \qquad and \qquad L^{p,p} = L^p.$$

Let us recall also the interpolation inequality (see [7]) which allows us to obtain some embeddings of spaces.

Lemma 2.1. *Let p_0, p_1, p, q in $[1,+\infty]$ and $0<\theta<1$.*

- *If $q\leq p$, then*
$$\big[L^p(L^{p_0}),L^p(L^{p_1})\big]_{(\theta,q)}\hookrightarrow L^p\big([L^{p_0},L^{p_1}]_{(\theta,q)}\big).$$
- *If $p\leq q$, then*
$$L^p\big([L^{p_0},L^{p_1}]_{(\theta,q)}\big)\hookrightarrow \big[L^p(L^{p_0}),L^p(L^{p_1})\big]_{(\theta,q)}.$$

Recall also the definition of Lebesgue anisotropic spaces. It notes $L^p_v(L^q_h)$ the space $L^p_v(\mathbb{R};\, L^q(\mathbb{R}^2))$ defined by the norm
$$\|f\|_{L^p_v(L^q_h)}:=\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^2}|f(x,y,z)|^q\,dxdy\Big)^{\frac{p}{q}}\,dz\Big)^{\frac{1}{p}}.$$
Similarly, we denote by $L^q_h(L^p_v)$ the space $L^q(\mathbb{R}^2;\, L^p(\mathbb{R}))$, with the norm
$$\|f\|_{L^q_h(L^p_v)}:=\Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}}|f(x,y,z)|^p\,dz\Big)^{\frac{q}{p}}\,dxdy\Big)^{\frac{1}{q}}.$$
In the cylindrical frame, $\omega=\nabla\times u$ admits only the component in the direction e_θ and in the cartesian frame two components:
$$\omega=(\omega^1,\omega^2,0)$$
with $\omega^1=\partial_y u^3-\partial_z u^2$ and $\omega^2=\partial_z u^1-\partial_x u^3$, u^j for $1\leq j\leq 3$ the components of u in the cartesian frame and (x,y,z) are the variables in this base. The fact that $u^\theta=0$, implies that in the cylindrical frame, we have:
$$u\cdot\nabla=u^r\partial_r+u^z\partial_z,$$
and
$$\operatorname{div}u=\partial_r u^r+\frac{u^r}{r}+\partial_z u^z.$$
We recall that, if u is a solution of (NS_v), then ω verifies the following equation
$$\partial_t\omega+(u^r\partial_r+u^z\partial_z)\omega-\frac{u^r}{r}\omega-\partial_z^2\omega=0,$$
and using that $u^\theta=0$, we obtain
$$\partial_t\omega+(u\cdot\nabla)\omega-\frac{u^r}{r}\omega-\partial_z^2\omega=0. \tag{2.1}$$
In other words, in the axisymmetric case, (NS_v) became a two dimensional problem. We recall also that in the two dimensional case, $\omega=\partial_x u^2-\partial_y u^1$, verifies the following transport-diffusion equation :
$$\partial_t\omega+(u\cdot\nabla)\omega-\partial_z^2\omega=0.$$
In the three dimensional space, in the axisymmetric case $\frac{\omega}{r}$ plays a similar role because we have
$$\partial_t\frac{\omega}{r}+(u\cdot\nabla)\frac{\omega}{r}-\partial_z^2\frac{\omega}{r}=0. \tag{2.2}$$

3. Proof of the theorem 1.1

3.1. **A prior estimates.** Using the equation (2.2) and the Biot-Savart law, we can control some important quantities in order to prove the global existence of the solution. More exactly, we have the following estimates.

Proposition 3.1. *Let u a axisymmetric solenoidal vector-field with vorticity $\omega=\omega^\theta e_\theta$. Let $(p,q,\lambda)\in[1,\infty]^3$, then we have*
$$u^r=\omega^\theta=0\quad\text{on the axis}\quad r=0.$$
The following inequalities :

- *If* $\frac{3}{2} \le p < \infty$ *such that* $\frac{1}{q} = \frac{1}{3} + \frac{1}{p}$, *then*

$$\|u\|_{L^{p,\lambda}} \lesssim \|\omega\|_{L^{q,\lambda}}, \qquad \|\frac{u^r}{r}\|_{L^{p,\lambda}} \lesssim \|\frac{\omega}{r}\|_{L^{q,\lambda}}, \qquad \|\partial_z u^r\|_{L^{p,\lambda}} \lesssim \|\partial_z \omega\|_{L^{q,\lambda}},$$

$$\|\partial_z u^z\|_{L^{p,\lambda}} \lesssim \|\partial_z \omega\|_{L^{q,\lambda}} \quad \textit{and} \quad \|\partial_z u^z\|_{L^{p,\lambda}} + \|\partial_r u^z\|_{L^{p,\lambda}} \lesssim \|\partial_r \omega\|_{L^{q,\lambda}} + \|\frac{\omega}{r}\|_{L^{q,\lambda}}.$$

- *If* $3 \le p < \infty$ *such that* $\frac{1}{q} = \frac{2}{3} + \frac{1}{p}$, *then*

$$\|u^r\|_{L^{p,\lambda}} \lesssim \|\partial_z \omega\|_{L^{q,\lambda}}, \qquad \|\frac{u^r}{r}\|_{L^{p,\lambda}} \lesssim \|\partial_z \frac{\omega}{r}\|_{L^{q,\lambda}}$$

$$\|u^z\|_{L^{p,\lambda}} \lesssim \|\partial_r \omega\|_{L^{q,\lambda}} + \|\frac{\omega}{r}\|_{L^{q,\lambda}}, \quad \|\partial_z u^z\|_{L^{p,\lambda}} \lesssim \|\partial_z \partial_r \omega\|_{L^{q,\lambda}} + \|\partial_z \frac{\omega}{r}\|_{L^{q,\lambda}}$$

and

$$\|\partial_r u^r\|_{L^{p,\lambda}} \lesssim \|\partial_z \partial_r \omega\|_{L^{q,\lambda}} + \|\partial_z \frac{\omega}{r}\|_{L^{q,\lambda}}.$$

- *In the limiting case, that is* $p = \infty$

$$\|u\|_{L^\infty} \lesssim \|\omega\|_{L^{3,1}}, \qquad \|u^r\|_{L^\infty} \lesssim \|\partial_z \omega\|_{L^{\frac{3}{2},1}}, \qquad \|\frac{u^r}{r}\|_{L^\infty} \lesssim \|\partial_z \frac{\omega}{r}\|_{L^{\frac{3}{2},1}}$$

$$\|u^z\|_{L^\infty} \lesssim \|\partial_r \omega\|_{L^{\frac{3}{2},1}} + \|\frac{\omega}{r}\|_{L^{\frac{3}{2},1}}, \quad \|\partial_z u^z\|_{L^\infty} \lesssim \|\partial_z \partial_r \omega\|_{L^{\frac{3}{2},1}} + \|\partial_z \frac{\omega}{r}\|_{L^{\frac{3}{2},1}}$$

and

$$\|\partial_r u^r\|_{L^\infty} \lesssim \|\partial_z \partial_r \omega\|_{L^{\frac{3}{2},1}} + \|\partial_z \frac{\omega}{r}\|_{L^{\frac{3}{2},1}}.$$

Proof. The first assertion can be deduced from the fact that that $u^\theta = 0$: indeed, using that

$$u^\theta = u \cdot e_\theta$$

we have

$$-yu^1 + xu^2 = 0. \tag{3.3}$$

Consequently $u^1 = 0$ (resp. $u^2 = 0$) on the plan $x = 0$ (resp. $y = 0$). For ω^θ, we use the fact that ω has only the component in the direction e_θ, which implies

$$x\omega^1 + y\omega^2 = 0,$$

and consequently ω^1 (resp. ω^2) is vanishing on the plan $x = 0$ (resp. $y = 0$). This proves the result. Using the Biot-Savart, we have

$$u(X) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{X - X'}{|X - X'|^3} \times \omega(X') dX', \tag{3.4}$$

with $X = (x, y, z)$ and $X' = (x', y', z')$, and finally we have

$$|u| \lesssim \frac{1}{|\cdot|^2} \star |\omega|,$$

on the other hand, by the definition of the Lorentz space (Definition 2.1), we have

$$\frac{1}{|X|^2} \in L^{\frac{3}{2},\infty}(\mathbb{R}^3)$$

so, by using the Proposition 2.1, we deduce

$$\|u\|_{L^{p,\lambda}} \lesssim \|\omega\|_{L^{\frac{3p}{3+p},\lambda}} \qquad \text{for } \tfrac{3}{2} \le p < \infty \qquad \text{and} \qquad \|u\|_{L^\infty} \lesssim \|\omega\|_{L^{3,1}}.$$

By the inequality (3.4), we have

$$u^1(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{z-z'}{|X-X'|^3}\,\omega^2(X')dX'$$

and

$$u^2 = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{z-z'}{|X-X'|^3}\,\omega^1(X')dX'$$

with $\omega^1(X') = -\sin\theta'\,\omega^\theta(X')$ and $\omega^2(X') = \cos\theta'\,\omega^\theta(X')$. Consequently we have

$$\begin{aligned} u^r(X) &= \cos\theta\, u^1(X) + \sin\theta\, u^2(X) \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{z-z'}{|X-X'|^3}\big\{-\cos\theta\cos\theta' - \sin\theta\sin\theta'\big\}\omega^\theta(X')dX' \end{aligned}$$

where we have denoted by (r,θ,z) the variables in the cylindrical frame, and we recall also that in this cylindrical frame we have $X = (r\cos\theta, r\sin\theta, z)$ and $X' = (r'\cos\theta', r'\sin\theta', z')$. We have

$$\begin{aligned} u^r(X) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{z-z'}{|X-X'|^3}\big\{\cos\theta\sin\theta' + \sin\theta\cos\theta'\big\}\omega^\theta(X')dX' \\ &= -\frac{1}{4\pi}\int_{\mathbb{R}_+}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{\mathbb{R}} \frac{z-z'}{|X-X'|^3}\cos(\theta-\theta')\omega^\theta(r',z')r'dr'd\theta'dz', \end{aligned}$$

on the other hand

$$\frac{z-z'}{|X-X'|^3} = \partial_{z'}\frac{1}{|X-X'|},$$

by integration by parts, we found

$$u^r(X) = \frac{1}{4\pi}\int_{\mathbb{R}_+}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{\mathbb{R}} \frac{1}{|X-X'|}\cos(\theta-\theta')\partial_{z'}\omega^\theta(r',z')r'dr'd\theta'dz'.$$

Using the fact that u^r does not depend in the variable θ (X=(r,0,z)), then

$$u^r(t,r,z) = \frac{1}{4\pi}\int_{\mathbb{R}_+}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{\mathbb{R}} \frac{1}{|X-X'|}\cos\theta'\partial_{z'}\omega^\theta(t,r',z')r'dr'd\theta'dz', \tag{3.5}$$

which implies that

$$|u^r| \lesssim \frac{1}{|\cdot|}\star|\partial_{z'}\omega|.$$

By the definition of Lorentz spaces, we have

$$\frac{1}{|X|^\sigma} \in L^{\frac{3}{\sigma},\infty}(\mathbb{R}^3), \quad \text{for } 0<\sigma<3$$

and so, by using the Proposition 2.1, we obtain the desired inequality.

Concerning the second inequality of the proposition, and thanks to the inequality (3.5), we have

$$|\partial_z u^r| \lesssim \frac{1}{|\cdot|^2}\star|\partial_{z'}\omega|,$$

and consequently, by using the Proposition 2.1, we obtain the desired inequality. For $\frac{u^r}{r}$, we use the identity (3.5)) and we follow the same computations as in [18], in order to obtain

$$u^r(t,r,z) = \frac{1}{4\pi}\int_{\mathbb{R}_+}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{\mathbb{R}} \frac{\cos\theta'\partial_{z'}\omega^\theta(t,r',z')}{(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{\frac{1}{2}}}\, r'dr'd\theta'dz'$$

$$= \frac{1}{4\pi}\int_{\mathbb{R}_+}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\mathbb{R}} \frac{\cos\theta'\partial_{z'}\omega^\theta(t,r',z')}{(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{\frac{1}{2}}}\, r'dr'd\theta'dz'$$

$$+ \frac{1}{4\pi}\int_{\mathbb{R}_+}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{\mathbb{R}} \frac{\cos\theta'\partial_{z'}\omega^\theta(t,r',z')}{(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{\frac{1}{2}}}\, r'dr'd\theta'dz'$$

for the second part, with the following change of variables $\theta' \to \theta' + \pi$, in order to obtain

$$\begin{aligned} u^r(t,r,z) &= \frac{1}{4\pi}\int_{\mathbb{R}_+}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\mathbb{R}} \frac{\cos\theta'\partial_{z'}\omega^\theta(t,r',z')}{(r^2+r'^2-2rr'\cos\theta'+(z-z')^2)^{\frac{1}{2}}}\, r'dr'd\theta'dz' \\ &- \frac{1}{4\pi}\int_{\mathbb{R}_+}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\mathbb{R}} \frac{\cos\theta'\partial_{z'}\omega^\theta(t,r',z')}{(r^2+r'^2+2rr'\cos\theta'+(z-z')^2)^{\frac{1}{2}}}\, r'dr'd\theta'dz'. \end{aligned} \tag{3.6}$$

If $|X-X'| \le r$, we use the inequality (3.5) and the fact that $r' \le 2r$, to obtain

$$\Big|\int_{|X-X'|\le r} \frac{\cos\theta'\partial_{z'}\omega^\theta(t,r',z')}{|X-X'|} r'dr'd\theta'dz'\Big| \lesssim r\int_{\mathbb{R}^3} \frac{1}{|X-X'|}\Big|\partial_{z'}\frac{\omega(t,X')}{r'}\Big|dX'.$$

If $|X-X'| \ge r$, we use the inequality (3.6) and the fact that

$$\Big|\Big(r^2+r'^2+2rr'\cos\theta'+(z-z')^2\Big)^{-\frac{1}{2}} - \Big(r^2+r'^2-2rr'\cos\theta'+(z-z')^2\Big)^{-\frac{1}{2}}\Big| \le \frac{2r}{|X-X'|^2},$$

because $-\frac{\pi}{2} \le \theta' \le \frac{\pi}{2}$. Consequently, in this region, we found

$$\begin{aligned} \Big|\int_{|X-X'|\ge r} \frac{\cos\theta'\partial_{z'}\omega^\theta(t,r',z')}{|X-X'|} r'dr'd\theta'dz'\Big| &\lesssim r\int_{|X-X'|\ge r} \frac{1}{|X-X'|^2}|\partial_{z'}\omega(t,X')|dX' \\ &\lesssim r\int_{|X-X'|\ge r} \frac{r'}{|X-X'|^2}\Big|\partial_{z'}\frac{\omega(t,X')}{r'}\Big|dX', \end{aligned}$$

and as, by using the fact that $r' = r' - r + r$ and $|r'-r| \le |X-X'|$, we obtain

$$\Big|\int_{|X-X'|\ge r} \frac{\cos\theta'\partial_{z'}\omega^\theta(t,r',z')}{|X-X'|} r'dr'd\theta'dz'\Big| \lesssim r\int_{\mathbb{R}^3} \frac{}{|X-X'|}\Big|\partial_{z'}\frac{\omega(t,X')}{r'}\Big|.$$

So

$$|u^r(t,X)| \lesssim r\int_{\mathbb{R}^3} \frac{1}{|X-X'|}\Big|\partial_{z'}\frac{\omega(t,X')}{r'}\Big|dX',$$

and we have also

$$|u^r(t,X)| \lesssim r\int_{\mathbb{R}^3} \frac{1}{|X-X'|^2}\Big|\frac{\omega(t,X')}{r'}\Big|dX'.$$

To conclude is enough now to use the convolution laws. Concerning u^z, by using the Biot-Savart law, we have

$$u^z(X) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-x')\omega^2(X') - (y-y')\omega^1(X')}{|X-X'|^3}\,dX'. \tag{3.7}$$

On the other hand

$$\frac{x-x'}{|X-X'|^3} = \partial_{x'}\frac{1}{|X-X'|} \quad \text{and} \quad -\frac{y-y'}{|X-X'|^3} = -\partial_{y'}\frac{1}{|X-X'|},$$

an so, by integration by parts, we obtain

$$u^z(X) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\partial_{y'}\omega^1 - \partial_{x'}\omega^2}{|X-X'|}\,dX'.$$

In the cylindrical variables, we have

$$\begin{aligned}
&\partial_{x'} = \cos\theta'\partial_{r'} - \frac{1}{r'}\sin\theta'\partial_{\theta'}, \quad \partial_{y'} = \sin\theta'\partial_{r'} + \frac{1}{r'}\cos\theta'\partial_{\theta'},\\
&\omega^1 = -\sin\theta'\omega^\theta \quad \text{and} \quad \omega^2 = \cos\theta'\omega^\theta,
\end{aligned}$$

and consequently

$$\begin{aligned}
\partial_{y'}\omega^1 - \partial_{x'}\omega^2 &= -\sin^2\theta'\partial_{r'}\omega^\theta - \frac{1}{r'}\cos^2\theta'\omega^\theta - \big(\cos^2\theta'\partial_{r'}\omega^\theta + \frac{1}{r'}\sin^2\theta'\omega^\theta\big)\\
&= -\partial_{r'}\omega^\theta - \frac{\omega^\theta}{r'},
\end{aligned}$$

and so

$$u^z(X) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|X-X'|}\big(\partial_{r'}\omega^\theta + \frac{\omega^\theta}{r'}\big)dX'. \tag{3.8}$$

So

$$|u^z| \lesssim \frac{1}{|\cdot|}\star\big(|\partial_{r'}\omega| + |\frac{\omega}{r'}|\big), \tag{3.9}$$

and in the same manner, for the derivative in the variable z, we use the same computations and thanks to the inequalities (3.7)) and (3.8), we found

$$|\partial_z u^z| \lesssim \begin{cases} \frac{1}{|\cdot|^2}\star|\partial_{z'}\omega| \\ \frac{1}{|\cdot|^2}\star\big(|\partial_{r'}\omega| + |\frac{\omega}{r'}|\big) \\ \frac{1}{|\cdot|}\star\big(|\partial_{z'}\partial_{r'}\omega| + |\partial_{z'}\frac{\omega}{r'}|\big). \end{cases} \tag{3.10}$$

Using the convolution laws, we deduce the desired inequalities. Concerning $\partial_r u^z$, we use the inequality (3.8), to obtain

$$\partial_r u^z(X) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{r - r'\cos\theta'}{|X-X'|^3}\big(\partial_{r'}\omega^\theta - \frac{\omega^\theta}{r'}\big)dX',$$

then

$$|\partial_r u^z| \lesssim \frac{1}{|\cdot|^2}\star\big(|\partial_{r'}\omega| + |\frac{\omega}{r'}|\big) \tag{3.11}$$

because

$$\Big|\frac{r - r'\cos\theta'}{|X-X'|}\Big| \le 1.$$

Finally, for $\partial_r u^r$, is enough to use the fact that

$$\operatorname{div} u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0.$$

This proves the proposition. $\square$

Conforming to the Proposition 3.1, we need to control ω in the Lorentz space $L^{\frac{3}{2},1}$, which is the goal of the following inequality. More precisely we give an estimate on the solution of the transport-diffusion equation.

Proposition 3.2. *Let $1 < p < 2$, $1 \leq q \leq p$, $\omega_0 \in L^{p,q}$ and u a regular axisymmetric vector field such that $\frac{u^r}{r} \in L^1_t(L^\infty)$ and $\operatorname{div} u = 0$. Let $\omega \in L^\infty_t(L^{p,q})$ and $\partial_z\omega \in L^2_t(L^{p,q})$ a solution for the following system*

$$(\mathrm{TD}_{\mathrm{mod}})\ \begin{cases} \partial_t\omega + (u\cdot\nabla)\omega - \partial_z^2\omega = \frac{u^r}{r}\omega \\ \omega_{|t=0} = \omega_0. \end{cases}$$

Then

$$\|\omega(t)\|_{L^{p,q}} + \|\partial_z\omega\|_{L^2_t(L^{p,q})} \lesssim \|\omega_0\|_{L^{p,q}} e^{\int_0^t \|\frac{u^r}{r}\|_{L^\infty}}.$$

Proof. The first step is to control ω in the Lebesgue spaces. Let $1 < p < \infty$, we multiply the equation verified by ω by $|\omega|^{p-1}\operatorname{sign}\omega$. After an integration by parts combined with the fact that div $u = 0$, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\omega\|^p_{L^p} + \frac{4(p-1)}{p^2}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^2_{L^2} = \int_{\mathbb{R}^3}\frac{u^r}{r}|\omega|^p dx,$$

and by using the Hölder inequality and the integration in the time variable, we obtain

$$\|\omega(t)\|^p_{L^p} + \frac{4(p-1)}{p}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^2_{L^2_t(L^2)} \leq \|\omega_0\|^p_{L^p} + p\int_0^t \|\frac{u^r}{r}(\tau)\|_{L^\infty}\|\omega(\tau)\|^p_{L^p}d\tau.$$

Finally, the Gronwall lemma implies that

$$\|\omega(t)\|^p_{L^p} + \frac{4(p-1)}{p}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^2_{L^2_t(L^2)} \leq \|\omega_0\|^p_{L^p}\exp\Big(p\int_0^t\|\frac{u^r}{r}(\tau)\|_{L^\infty}d\tau\Big). \tag{3.12}$$

In order to estimate $\partial_z\omega$ in L^p we will use the following Lemma. We postponed the proof of this lemma for the moment.

Lemma 3.1. *Let $1 \leq p \leq 2$ et $f \in L^p(\mathbb{R}^N)$ such that $\partial_i|u|^{\frac{p}{2}} \in L^2(\mathbb{R}^N)$. Then*

$$\|\partial_i f\|_{L^p} \lesssim \Big\|\partial_i|f|^{\frac{p}{2}}\Big\|_{L^2}\|f\|^{\frac{2-p}{2}}_{L^p}.$$

For $p \leq 2$, using the Lemma 3.1 and the inequality (3.12), we obtain that

$$\begin{aligned}\|\partial_z\omega\|_{L^2_t(L^p)} &\lesssim \Big(\int_0^t\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^2_{L^2}\|\omega\|^{2-p}_{L^p}d\tau\Big)^{\frac{1}{2}}\\ &\lesssim \|\omega\|^{\frac{2-p}{2}}_{L^\infty_t(L^p)}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L^2_t(L^2)}\\ &\lesssim \|\omega_0\|_{L^p}\exp\Big(\int_0^t\|\frac{u^r}{r}(\tau)\|_{L^\infty}d\tau\Big).\end{aligned}$$

So

$$\|\omega(t)\|_{L^p} + \|\partial_z\omega\|_{L^2_t(L^p)} \lesssim \|\omega_0\|_{L^p}\exp\Big(\int_0^t\|\frac{u^r}{r}(\tau)\|_{L^\infty}d\tau\Big). \tag{3.13}$$

We denote by $\mathcal{T}$ and $\mathcal{S}$ the following linear operators:

$$\mathcal{T}:\quad \begin{aligned} L^p &\longrightarrow L^p\\ \omega_0 &\longmapsto \omega\end{aligned}\qquad\qquad \mathcal{S}:\quad \begin{aligned} L^p &\longrightarrow L^2_t(L^p)\\ \omega_0 &\longmapsto \partial_z\omega,\end{aligned}$$

with ω solution of the system ($\mathrm{TD}_{\mathrm{mod}}$). By definition, we have $\mathcal{T}$ and $\mathcal{S}$ are linear operators, then by Lemma 2.1, we obtain

$$\|\omega(t)\|_{L^{p,q}} + \|\partial_z\omega(\tau)\|_{L^2_t(L^{p,q})} \lesssim \|\omega_0\|_{L^{p,q}} \exp\Big(\int_0^t \|\frac{u^r}{r}(\tau)\|_{L^\infty} d\tau\Big). \tag{3.14}$$

This proves the proposition. □

Using the same computations, we obtain the following corollary.

Corollary 3.1. *Let $1 < p < 2$, $1 \le q \le p$, $r^{-1}\omega_0 \in L^{p,q}$ and u a regular axisymmetric vector field such that* $\operatorname{div} u = 0$. *Let $r^{-1}\omega \in L^\infty_t(L^{p,q})$ and $r^{-1}\partial_z\omega \in L^2_t(L^{p,q})$ a solution for the following system*

$$\begin{cases} \partial_t \frac{\omega}{r} + (u\cdot\nabla)\frac{\omega}{r} - \partial_z^2 \frac{\omega}{r} = 0 \\ \frac{\omega}{r}_{|t=0} = \frac{\omega_0}{r}. \end{cases}$$

Then

$$\Big\|\frac{\omega}{r}(t)\Big\|_{L^{p,q}} + \Big\|\partial_z\frac{\omega}{r}\Big\|_{L^2_t(L^{p,q})} \lesssim \Big\|\frac{\omega_0}{r}\Big\|_{L^{p,q}}.$$

Remark 3.1. *Using the inequality* (3.12) *and the fact that*

$$\|\frac{\omega}{r}(t)\|_{L^p} \le \|\frac{\omega_0}{r}\|_{L^p},$$

we deduce thanks to [3], *that* $\forall (p,q) \in]1,\infty[\times[1,\infty]$

$$\|\omega(t)\|_{L^{p,q}} \le \|\omega_0\|_{L^{p,q}} e^{\int_0^t \|\frac{u^r}{r}(\tau)\|_{L^\infty} d\tau}$$

and

$$\|\frac{\omega}{r}(t)\|_{L^{p,q}} \le \|\frac{\omega_0}{r}\|_{L^{p,q}}.$$

Using the Proposition 3.1, the Corollary 3.1 and the Hölder's inequality, we have

$$\begin{aligned} \Big\|\frac{u^r}{r}\Big\|_{L^1_t(L^\infty)} \lesssim \Big\|\partial_z\frac{\omega}{r}\Big\|_{L^1_t(L^{\frac32,1})} &\lesssim t^{\frac12}\Big\|\partial_z\frac{\omega}{r}\Big\|_{L^2_t(L^{\frac32,1})} \\ &\lesssim t^{\frac12}\Big\|\frac{\omega_0}{r}\Big\|_{L^{\frac32,1}}. \end{aligned} \tag{3.15}$$

Consequently, for all $p \in]1,2[$ and $q \in [1,p]$, the inequalities (3.14) and (3.15), imply

$$\|\omega(t)\|_{L^{p,q}} + \|\partial_z\omega\|_{L^2_t(L^{p,q})} \le C\|\omega_0\|_{L^{p,q}} e^{Ct^{\frac12}\|\frac{\omega_0}{r}\|_{L^{\frac32,1}}}. \tag{3.16}$$

So, the Proposition 3.1, Remark 3.1 and the inequality (3.15), implies that $(p,q) \in (\frac32,\infty)\times[1,\infty]$,

$$\|u(t)\|_{L^{p,q}} \le C\|\omega_0\|_{L^{\frac{3p}{3+p},q}} e^{Ct^{\frac12}\|\frac{\omega_0}{r}\|_{L^{\frac32,1}}}.$$

So, if $\omega \in L^{\frac32,1}$, then, the previous inequalities imply $u \in L^{3,1}$, which is embedded in the dual space of $L^{\frac32,1}$. Consequently, thanks to the Proposition II.1 in [9] and by using the equation verified by ω (2.1), we deduce the following result of the existence of the solution.

Corollary 3.2. *Let $\omega_0^\theta \in L^{\frac32,1}(\mathbb{R}^3)$ an axisymmetric function, such that $\frac{\omega_0^\theta}{r} \in L^{\frac32,1}(\mathbb{R}^3)$. Let u_0 the axisymmetric vector field such that* $\operatorname{div} u_0 = 0$ *and with the vorticity $\omega_0 = \omega_0^\theta(r,z)e_\theta$, which is given by the Biot-Savart law :*

$$u_0(X) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{X-Y}{|X-Y|^3} \times \omega_0(Y)\, dY.$$

Then, the system $(\mathrm{NS_v})$ *admits a global in time solution* u *such that the vorticity* ω *satisfies*

$$\begin{aligned}
&\omega \in \mathscr{C}\big(\mathbb{R}_+;\, L^{\frac32,1}(\mathbb{R}^3)\big), &&\partial_z\omega \in L^2_{loc}\big(\mathbb{R}_+;\, L^{\frac32,1}(\mathbb{R}^3)\big)\\
&\frac{\omega}{r} \in \mathscr{C}\big(\mathbb{R}_+;\, L^{\frac32,1}(\mathbb{R}^3)\big), &&\partial_z\frac{\omega}{r} \in L^2_{loc}\big(\mathbb{R}_+;\, L^{\frac32,1}(\mathbb{R}^3)\big).
\end{aligned}$$

Moreover, for all $t \geq 0$, *we have*

$$\|\omega(t)\|_{L^{\frac32,1}} + \|\partial_z\omega\|_{L^2_t(L^{\frac32,1})} \leq C\|\omega_0\|_{L^{\frac32,1}} e^{Ct^{\frac12}\|r^{-1}\omega_0\|_{L^{\frac32,1}}}$$

and

$$\|r^{-1}\omega(t)\|_{L^{\frac32,1}} + \|r^{-1}\partial_z\omega\|_{L^2_t(L^{\frac32,1})} \leq C\|r^{-1}\omega_0\|_{L^{\frac32,1}}.$$

Proof of Lemma 3.1.
Let us remark that

$$\|\partial_i f\|_{L^p} = \|\partial_i|f|\|_{L^p} \qquad \text{et} \qquad |f| = |f|^{\frac{p}{2}\frac{2}{p}},$$

and so, we have

$$\partial_i|f| = \frac{p}{2}\partial_i(|f|^{\frac{p}{2}})|f|^{\frac{2-p}{2}}.$$

The Hölder's inequality, implies that

$$\|\partial_i u\|_{L^p} \lesssim \Big\|\partial_i|u|^{\frac{p}{2}}\Big\|_{L^2}\|u\|_{L^p}^{\frac{2-p}{2}}.$$

This proves the Lemma. □

3.2. **Uniqueness.** In order to prove the uniqueness of the solution for the system $(\mathrm{NS_v})$, it will be enough to prove the uniqueness for the equation (2.1). Let ω_1 and ω_2 two solutions, and let define $\delta\omega = \omega_2 - \omega_1$ their differences, which verifies the following system :

$$\begin{cases}\partial_t\delta\omega + (u_2\cdot\nabla)\delta\omega - \partial_z^2\delta\omega = -(\delta u\cdot\nabla)\omega_1 + \frac{u_2^r}{r}\delta\omega + \frac{\delta u^r}{r}\omega_1\\ \delta\omega_{|t=0} = 0.\end{cases}$$

The functional framework where we control the differences of the two solutions is L^p with $\frac65 \leq p < \frac32$. Let us admit for the moment the following Lemma the proof of which is postponed.

Lemma 3.2. *Let* ω_i *with* $1 \leq i \leq 2$ *two solutions of the equation* (2.1) *with the same initial data. Let us suppose that for* $i = 1, 2$ *we have*

$$\omega_i \in L^\infty_t(L^{\frac32,1}), \quad \partial_z\omega_i \in L^2_t(L^{\frac32,1}) \quad \partial_r\omega_i \in L^\infty_t(L^{\frac32}) \quad \textit{and} \quad \partial_z\partial_r\omega_i \in L^2_t(L^{\frac32})$$

Then

$$\delta\omega \in L^\infty_t(L^p) \qquad \textit{and} \qquad \partial_z|\delta\omega|^{\frac{p}{2}} \in L^2_t(L^2).$$

The energy estimates imply that

$$\begin{aligned}\frac1p\frac{d}{dt}\|\delta\omega\|^p_{L^p} + \frac{4(p-1)}{p^2}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^2_{L^2} \leq \|\frac{u_2^r}{r}\|_{L^\infty}\|\delta\omega\|^p_{L^p} + \|\frac{\omega_1\delta u^r}{r}\|_{L^p}\|\delta\omega\|^{p-1}_{L^p}\\ + \|(\delta u\cdot\nabla)\omega_1\|_{L^p}\|\delta\omega\|^{p-1}_{L^p}.\end{aligned}$$

Using Hölder inequality, Sobolev embedding, Proposition 3.1 and Lemma 3.1, we have

$$
\begin{aligned}
\|\frac{\omega_1\delta u^r}{r}\|_{L^p}+\|(\delta u\cdot\nabla)\omega_1\|_{L^p} &\leq \big(\|\frac{\omega_1}{r}\|_{L^{\frac32}}+\|\partial_r\omega_1\|_{L^{\frac32}}\big)\|\delta u^r\|_{L^{\frac{3p}{3-2p}}}\\
&+\|\partial_z\omega_1\|_{L^6_h(L_v^{\frac32})}\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}\\
&\lesssim \big(\|\frac{\omega_1}{r}\|_{L^{\frac32}}+\|\partial_r\omega_1\|_{L^{\frac32}}\big)\|\partial_z\delta\omega\|_{L^p}+\|\partial_z\partial_r\omega_1\|_{L^{\frac32}}\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}\\
&\lesssim \Big(\|\frac{\omega_1}{r}\|_{L^{\frac32}}+\|\partial_r\omega_1\|_{L^{\frac32}}\Big)\|\partial_z|\delta\omega|^{\frac p2}\|_{L^2}\|\delta\omega\|_{L^p}^{\frac{2-p}{2}}+\|\partial_z\partial_r\omega_1\|_{L^{\frac32}}\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}.
\end{aligned}
$$

Concerning $\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}$ and using the fact that

$$\Delta\delta u^z=\partial_x^2\delta u^z+\partial_y^2\delta u^z+\partial_z^2\delta u^z=\partial_r\delta\omega+\frac{\delta\omega}{r},$$

we obtain by integration by parts that

$$|\delta u^z|\lesssim\frac{1}{|\cdot|^2}\star|\delta\omega|,$$

Then, using the convolution laws, we obtain

$$
\begin{aligned}
\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})} &\lesssim \|\delta\omega\|_{L_h^{\frac{6p}{6-p},\frac{6p}{6-p}}(L_v^p)}\\
&\lesssim \|\delta\omega\|_{L^p}.
\end{aligned}
$$

The Young inequality, implies that

$$\frac{d}{dt}\|\delta\omega\|_{L^p}^p\leq\Big(\|\frac{u_2^r}{r}\|_{L^\infty}+\|\frac{\omega_1}{r}\|^2_{L^{\frac32}}+\|\partial_r\omega_1\|^2_{L^{\frac32}}+\|\partial_z\partial_r\omega_1\|_{L^{\frac32}}\Big)\|\delta\omega\|^p_{L^p}.$$

So, we obtain the uniqueness of the solution if $\partial_r\omega_1\in L_t^\infty(L^{\frac32})$ and $\partial_z\partial_r\omega_1\in L_t^2(L^{\frac32})$ because the inequality (3.15) and the Corollary 3.1, imply $\big(\|\frac{u_2^r}{r}\|_{L^\infty}+\|\frac{\omega_1}{r}\|^2_{L^{\frac32}}\big)\in L_t^1$.

The first step is to prove that $\partial_r\omega_1\in L_t^\infty(L^{\frac32})$. More precisely we prove that we can propagate the regularity of $\partial_r\omega$ in the Lorentz space $L^{\frac32}$ and moreover, we prove that we have a regularizing effect in this space.

4. Propagation of the regularity $\partial_r\omega$

Proposition 4.1. *Let $\omega_0\in L^{\frac32,1}\cap L^{3,1}$ such that $\omega_0/r\in L^{\frac32,1}$ and $\partial_r\omega_0\in L^{\frac32}$. Let $\partial_r\omega\in L_t^\infty(L^{\frac32})$, $\partial_z\partial_r\omega\in L_t^2(L^{\frac32})$ a solution of the following system*

$$\begin{cases}\partial_t\partial_r\omega+(u\cdot\nabla)\partial_r\omega-\partial_z^2\partial_r\omega=-\frac{u^r}{r}\frac{\omega}{r}+\partial_ru^r\frac{\omega}{r}+\frac{u^r}{r}\partial_r\omega-\partial_ru^r\partial_r\omega-\partial_ru^z\partial_z\omega\\ \partial_r\omega_{|t=0}=\partial_r\omega_0.\end{cases}$$

Then

$$\|\partial_r\omega(t)\|_{L^{\frac32}}+\|\partial_z\partial_r\omega\|_{L_t^2(L^{\frac32})}\leq C(t,\omega_0).$$

Proof. First note that the fact that $\partial_ru^r=-\frac{u^r}{r}-\partial_zu^z$ and $\partial_ru^z=\partial_zu^r-\omega$, then we deduce the following equation

$$
\begin{aligned}
\partial_t\partial_r\omega+(u\cdot\nabla)\partial_r\omega-\partial_z^2\partial_r\omega &= 2\frac{u^r}{r}(\partial_r\omega-\frac{\omega}{r})-\partial_zu^z\frac{\omega}{r}+\partial_zu^z\partial_r\omega-\partial_zu^r\partial_z\omega+\omega\partial_z\omega\\
&=2\frac{u^r}{r}(\partial_r\omega-\frac{\omega}{r})+\partial_zu^z\partial_r\omega-\partial_zu^z\frac{\omega}{r}+g
\end{aligned}
$$

with

$$g=-\partial_zu^r\partial_z\omega+\omega\partial_z\omega. \tag{4.17}$$

Multiplying the equation verified by $\partial_r\omega$ by $|\partial_r\omega|^{\frac12}$ and integrating in space, we obtain

$$\frac23\frac{d}{dt}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\frac89\|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2}\le 2\|\frac{u^r}{r}\|_{L^\infty}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\int\partial_z u^z|\partial_r\omega|^{\frac32}$$
$$+\big(2\|\frac{u^r}{r}\|_{L^\infty}\|\frac{\omega}{r}\|_{L^{\frac32}}+\|\partial_z u^z\frac{\omega}{r}\|_{L^{\frac32}}+\|g\|_{L^{\frac32}}\big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}.$$

Integrating by parts and using the Cauchy-Schwartz inequality, we have

$$\int\partial_z u^z|\partial_r\omega|^{\frac32}=-2\int u^z|\partial_r\omega|^{\frac34}\partial_z|\partial_r\omega|^{\frac34}\le 2\|u^z\|_{L^\infty}\|\partial_z|\partial_r\omega|^{\frac34}\|_{L^2}\|\partial_r\omega\|^{\frac34}_{L^{\frac32}}.$$

And finally

$$\begin{aligned}\frac{d}{dt}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2}&\lesssim\Big(\|\frac{u^r}{r}\|_{L^\infty}+\|u^z\|^2_{L^\infty}\Big)\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}\\&+\Big(\|\partial_z u^z\frac{\omega}{r}\|_{L^{\frac32}}+\|\frac{u^r}{r}\|_{L^\infty}\|\frac{\omega}{r}\|_{L^{\frac32}}+\|g\|_{L^{\frac32}}\Big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}.\end{aligned}\tag{4.18}$$

By Hölder, (3.10) inequalities and interpolation, we have

$$\begin{aligned}\|\partial_z u^z\frac{\omega}{r}\|_{L^{\frac32}}&\le\|\frac{\omega}{r}\|_{L^{\frac32}_h(L^\infty_v)}\|\partial_z u^z\|_{L^\infty_h(L^{\frac32}_v)}\lesssim\|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac23}_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}\big(\|\partial_z\partial_r\omega\|^{\frac13}_{L^{\frac32}}+\|\partial_z\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}}\big)\\&\lesssim\|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}+\|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac23}_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}\|\partial_z\partial_r\omega\|^{\frac13}_{L^{\frac32}},\end{aligned}$$

and consequently by Lemma 3.1 and Hölder inequality, we obtain

$$\begin{aligned}\|\partial_z u^z\frac{\omega}{r}\|_{L^{\frac32}}\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}&\lesssim\|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}\\&+\|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac23}_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}\|\partial_z|\partial_r\omega|^{\frac34}\|^{\frac13}_{L^2}\|\partial_r\omega\|^{\frac{7}{12}}_{L^{\frac32}}\\&\le\varepsilon\|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2}+C\|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}\\&+C_\varepsilon\|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac45}_{L^{\frac32}}\|\partial_z\omega\|^{\frac45}_{L^{\frac32}}\|\partial_r\omega\|^{\frac{7}{10}}_{L^{\frac32}}.\end{aligned}$$

Thus in view of (4.18) and the preceding inequality, we conclude

$$\begin{aligned}\frac{d}{dt}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2}&\lesssim\big(\|\frac{u^r}{r}\|_{L^\infty}+\|u^z\|^2_{L^\infty}\big)\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\big(\|\frac{u^r}{r}\|_{L^\infty}\|\frac{\omega}{r}\|_{L^{\frac32}}+\|g\|_{L^{\frac32}}\big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}\\&+\|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac45}_{L^{\frac32}}\|\partial_z\omega\|^{\frac45}_{L^{\frac32}}\|\partial_r\omega\|^{\frac{7}{10}}_{L^{\frac32}}\\&\lesssim\Big(\|\frac{u^r}{r}\|_{L^\infty}+\|u^z\|^2_{L^\infty}+\|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac45}_{L^{\frac32}}\|\partial_z\omega\|^{\frac45}_{L^{\frac32}}\Big)\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}\\&+\Big(\|\frac{u^r}{r}\|_{L^\infty}\|\frac{\omega}{r}\|_{L^{\frac32}}+\|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac45}_{L^{\frac32}}\|\partial_z\omega\|^{\frac45}_{L^{\frac32}}+\|g\|_{L^{\frac32}}\Big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}.\end{aligned}$$

Then Gronwall lemma, implies that

$$\begin{aligned}\|\partial_r\omega(t)\|_{L^{\frac32}}+\|\partial_z|\partial_r\omega|^{\frac34}\|^{\frac43}_{L^2_t(L^2)}&\le Ce^{C\int_0^t\big(\|\frac{u^r}{r}\|_{L^\infty}+\|u^z\|^2_{L^\infty}+\|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac45}_{L^{\frac32}}\|\partial_z\omega\|^{\frac45}_{L^{\frac32}}\big)d\tau}\\&\times\Big(\|\partial_r\omega_0\|_{L^{\frac32}}+\int_0^t\big(\|\frac{u^r}{r}\|_{L^\infty}\|\frac{\omega}{r}\|_{L^{\frac32}}+\|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac45}_{L^{\frac32}}\|\partial_z\omega\|^{\frac45}_{L^{\frac32}}+\|g\|_{L^{\frac32}}\big)d\tau\Big),\end{aligned}$$

finally the Lemma 3.1 and the above inequality assures that

$$\begin{aligned}\|\partial_r\omega(t)\|_{L^{\frac32}}+\|\partial_z\partial_r\omega\|_{L^2_t(L^{\frac32})}&\leq Ce^{C\int_0^t\left(\|\frac{u^r}{r}\|_{L^\infty}+\|u^z\|^2_{L^\infty}+\|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac45}_{L^{\frac32}}\|\partial_z\omega\|^{\frac45}_{L^{\frac32}}\right)d\tau}\\ &\times\Big(\|\partial_r\omega_0\|_{L^{\frac32}}+\int_0^t(\|\frac{u^r}{r}\|_{L^\infty}\|\frac{\omega}{r}\|_{L^{\frac32}}+\|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}}\|\partial_z\frac{\omega}{r}\|^{\frac45}_{L^{\frac32}}\|\partial_z\omega\|^{\frac45}_{L^{\frac32}}+\|g\|_{L^{\frac32}})d\tau\Big).\end{aligned}\tag{4.19}$$

Recalling that

$$\|\omega(t)\|_{L^{\frac32,1}}+\|\partial_z\omega\|_{L^2_t(L^{\frac32,1})}\leq C\|\omega_0\|_{L^{\frac32,1}}\exp\big(Ct^{\frac12}\|r^{-1}\omega_0\|_{L^{\frac32,1}}\big),$$

$$\|\frac{u^r}{r}\|_{L^1_t(L^\infty)}\lesssim\sqrt{t}\|\frac{\omega_0}{r}\|_{L^{\frac32,1}},\qquad \|\frac{\omega}{r}\|_{L^{\frac32,1}}+\|\partial_z\frac{\omega}{r}\|_{L^2_t(L^{\frac32,1})}\lesssim\|\frac{\omega_0}{r}\|_{L^{\frac32,1}}$$

and

$$\|\omega(t)\|_{L^{\frac32}}+\|\partial_z\omega\|_{L^2_t(L^{\frac32})}\lesssim\|\omega_0\|_{L^{\frac32}}e^{C\sqrt{t}\|\frac{\omega_0}{r}\|_{L^{\frac32,1}}}.$$

So, thanks to Hölder inequality and Proposition 3.1, we have

$$\int_0^t\|\partial_zu^r\partial_z\omega\|_{L^{\frac32}}\leq\int_0^t\|\partial_zu^r\|_{L^6}\|\partial_z\omega\|_{L^2}\lesssim\int_0^t\|\partial_z\omega\|^2_{L^2},$$

and consequently, the inequalities (3.13) and (3.15), imply

$$\int_0^t\|\partial_zu^r\partial_z\omega\|_{L^{\frac32}}\lesssim\|\omega_0\|^2_{L^2}e^{Ct^{\frac12}\|\frac{\omega_0}{r}\|_{L^{\frac32,1}}}.\tag{4.20}$$

Concerning $\|\partial_z\omega^2\|_{L^1_t(L^{\frac32,1})}$ the Proposition 2.1, implies that

$$\begin{aligned}\int_0^t\|\partial_z\omega^2\|_{L^{\frac32}}&\lesssim\int_0^t\||\omega|^{\frac12}\|_{L^6}\|\partial_z|\omega|^{\frac32}\|_{L^2}\\ &\lesssim t^{\frac12}\|\omega\|^{\frac12}_{L^\infty_t(L^3)}\|\partial_z|\omega|^{\frac32}\|_{L^2_t(L^2)},\end{aligned}$$

and so, the inequalities (3.12) and (3.15) as well as the Remark 3.1, imply that

$$\int_0^t\|\partial_z\omega^2\|_{L^{\frac32}}\lesssim t^{\frac12}\|\omega_0\|^2_{L^3}e^{Ct^{\frac12}\|\omega_0/r\|_{L^{\frac32,1}}}.\tag{4.21}$$

Concerning the $\|u^z\|^2_{L^2_t(L^\infty)}$ the Proposition 3.1 and the Remark 3.1 imply

$$\int_0^t\|u^z\|^2_{L^\infty}\lesssim\int_0^t\|\omega\|^2_{L^{3,1}}\lesssim t\|\omega_0\|^2_{L^{3,1}}e^{Ct^{\frac12}\|\frac{\omega_0}{r}\|_{L^{\frac32,1}}}.\tag{4.22}$$

Then we deduce from inequalities (4.19), (4.20), (4.21) and(4.22) that

$$\|\partial_r\omega(t)\|_{L^{\frac32}}+\|\partial_z\partial_r\omega\|_{L^2_t(L^{\frac32})}\leq C(t,\omega_0).$$

This completes the proof. □

Proof of Lemma 3.2.
In order to prove that $\delta\omega\in L^\infty_t(L^p)$ and $\partial_z|\omega|^{\frac p2}\in L^2_t(L^2)$ it is enough to prove that $(u_2\cdot\nabla)\delta\omega+$

$(\delta u\cdot\nabla)\omega_1-\frac{u_2^r}{r}\delta\omega-\frac{\delta u^r}{r}\omega_1\in L^1_t(L^p)$ for $p\le\frac32$. Using Hölder inequality and interpolation 3.1 (see [20]), we obtain

$$\begin{aligned}
\|(u_2\cdot\nabla)\delta\omega\|_{L^p} &\le \|u_2\|_{L^{\frac{3p}{3-2p}}}\sum_{i=1}^2(\|\partial_r\omega_i\|_{L^{\frac32}}+(\|\partial_z\omega_i\|_{L^{\frac32}})\\
&\le \|u_2\|_{L^3}^{\frac{3-2p}{p}}\|u_2\|_{L^\infty}^{\frac{3(p-1)}{p}}\sum_{i=1}^2(\|\partial_r\omega_i\|_{L^{\frac32}}+\|\partial_z\omega_i\|_{L^{\frac32}})\\
&\lesssim \|\omega_2\|_{L^{\frac32}}^{\frac{3-2p}{p}}\|\omega_2\|_{L^{3,1}}^{\frac{3(p-1)}{p}}\sum_{i=1}^2(\|\partial_r\omega_i\|_{L^{\frac32}}+\|\partial_z\omega_i\|_{L^{\frac32}})\\
&\lesssim \|\omega_2\|_{L^{\frac32}}^{\frac{3-2p}{p}}\big(\|\partial_r\omega_2\|_{L^{\frac32,1}}+\|\frac{\omega_2}{r}\|_{L^{\frac32,1}}+\|\partial_z\omega_2\|_{L^{\frac32,1}}\big)^{\frac{3(p-1)}{p}}\\
&\times\sum_{i=1}^2(\|\partial_r\omega_i\|_{L^{\frac32}}+\|\partial_z\omega_i\|_{L^{\frac32}})
\end{aligned}$$

and consequently, the above two propositions combined with the Corollary 3.2, imply $(u_2\cdot\nabla)\delta\omega\in L^1_t(L^p)$, and so the same computations give $(\delta u\cdot\nabla)\omega_1\in L^1_t(L^p)$. For $\frac{u_2^r}{r}\delta\omega$ and thanks to Hölder inequality, and by interpolation and by Proposition 3.1, we obtain

$$\begin{aligned}
\|\frac{u_2^r}{r}\delta\omega\|_{L^p}\le\|u_2^r\|_{L^{\frac{3p}{3-2p}}}\|\frac{\delta\omega}{r}\|_{L^{\frac32}} &\le\sum_{i=1}^2\|\frac{\omega_i}{r}\|_{L^{\frac32}}\|u_2^r\|_{L^3}^{\frac{3-2p}{p}}\|u_2^r\|_{L^\infty}^{\frac{3(p-1)}{p}}\\
&\lesssim\sum_{i=1}^2\|\frac{\omega_i}{r}\|_{L^{\frac32}}\|\omega_2\|_{L^{\frac32}}^{\frac{3-2p}{p}}\|\partial_z\omega_2\|_{L^{\frac32,1}}^{\frac{3(p-1)}{p}}.
\end{aligned}$$

And consequently, the Corollary 3.2 and the fact that $\frac{3(p-1)}{p}\le 2$ imply $\frac{u_2^r}{r}\delta\omega\in L^1_t(L^p)$ the same computation gives $\frac{\delta u^r}{r}\omega_1\in L^1_t(L^p)$. This proves the Lemma. $\square$

4.1. **Existence for less regular initial data.** In this part we prove the Theorem 1.2 on the existence of solutions for less regular initial data. In order to obtain this, we have to take into account more anisotropic estimates on $\frac{u^r}{r}$. We have, for all $1<p\le\frac32$, the following inequalities

$$\|\frac{u^r}{r}\|_{L^\infty_h(L_v^{\frac{p}{3-2p}})}\le C\|\partial_z\frac{\omega}{r}\|_{L^{p,1}}.$$

Indeed, by the estimates of the Proposition 3.1 we have

$$|\frac{u^r}{r}|\lesssim\frac{1}{|X|}\star|\partial_z\frac{\omega}{r}|.$$

So

$$\|\frac{u^r}{r}\|_{L^\infty_h}\lesssim\|\frac{1}{\sqrt{|X_h|^2+z^2}}\|_{L_h^{p'}}\star\|\partial_z\frac{\omega}{r}\|_{L^p_h}$$

Using the fact that the primitive of $r(r^2+z^2)^{-\frac{p'}{2}}$ is $\sqrt{r^2+z^2}^{2-p'}$ up to a constant, we obtain

$$\|\frac{u^r}{r}\|_{L^\infty_h}\lesssim\frac{1}{|z|^{\frac2p-1}}\star\|\partial_z\frac{\omega}{r}\|_{L^p_h}.$$

We take now the norm $L^{\frac{p}{3-2p}}$ in the vertical variable in order to obtain

$$\|\frac{u^r}{r}\|_{L^\infty_h(L_v^{\frac{p}{3-2p}})}\le C\|\partial_z\frac{\omega}{r}\|_{L^{p,1}}.$$

We can in this manner control the norm of ω in all L^p, we recall that ω verifies the following inequality

$$\partial_t\omega + u\nabla\omega - \frac{u^r}{r}\omega - \partial_z^2\omega = 0$$

So for $1 < p \leq 3/2$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\||\omega(t)|^{p/2}\|_{L^2}^2 + \|\partial_z(|\omega|^{p/2})\|_{L^2}^2 &\leq \int |\frac{u^r}{r}||\omega|^{p/2}|\omega|^{p/2}\\ &\leq \|\frac{u^r}{r}\|_{L_h^\infty L_v^{\frac{p}{3-2p}}}\||\omega|^{p/2}\|^2_{L_h^2(L_v^{\frac{2p}{3(p-1)}})}.\end{aligned}$$

As $H^s(\mathbb{R}_v) \subset L^{\frac{2p}{3p-3}}(\mathbb{R}_v)$ for $s = (3-2p)/(2p)$, then

$$\||\omega|^{p/2}\|^2_{L_h^2(L_v^{\frac{2p}{3p-3}})} \leq \||\omega|^{p/2}\|_{L^2}^{(4p-3)/p}\|\partial_z(|\omega|^{p/2})\|_{L^2}^{(3-2p)/p}.$$

So

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\||\omega(t)|^{p/2}\|_{L^2}^2 + \|\partial_z(|\omega|^{p/2})\|_{L^2}^2 &\leq \frac{1}{2}\|\frac{u^r}{r}\|^{\frac{2p}{4p-3}}_{L_h^\infty(L_v^{\frac{p}{3-2p}})}\||\omega|^{p/2}\|_{L^2}^2 + \frac{1}{2}\|\partial_z(|\omega|^{p/2})\|_{L^2}^2\\ &\leq C\|\partial_z\frac{\omega}{r}\|^{\frac{2p}{4p-3}}_{L^{p,1}}\||\omega|^{p/2}\|_{L^2}^2 + \frac{1}{2}\|\partial_z(|\omega|^{p/2})\|_{L^2}^2.\end{aligned}$$

By Gronwall lemma and taking in account that $\|\partial_z\frac{\omega}{r}\|_{L_t^{\frac{2p}{4p-3}}(L^{p,1})} \leq t^{\frac{3(p-1)}{4p-3}}\|\frac{\omega_0}{r}\|_{L^{p,1}}$, we obtain

$$\|\omega\|_{L^p} + \|\partial_z\omega\|_{L_t^2(L^p)} \leq \|\omega_0\|_{L^p}\exp(Ct^{\frac{3(p-1)}{4p-3}}\|\frac{\omega_0}{r}\|_{L^{p,1}}),$$

and by interpolation

$$\|\omega\|_{L^{p,1}} + \|\partial_z\omega\|_{L_t^2(L^{p,1})} \leq \|\omega_0\|_{L^{p,1}}\exp(Ct^{\frac{3(p-1)}{4p-3}}\|\frac{\omega_0}{r}\|_{L^{p,1}}).$$

In particular, the above inequality is valid for $p = 6/5$, and so we can prove the global existence of a solution in the case where $\frac{\omega}{r} \in L^{\frac{6}{5}+,1}$ and $\omega_0 \in L^{\frac{6}{5}+,1}$. First of all, we note that $\omega_0 \in L^{\frac{6}{5},1}$ which implies that $u_0 \in L^2$ and by energy estimates, we have

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\partial_z u\|_{L^2}^2 \leq \|u_0\|_{L^2}^2.$$

On the other hand, as $\omega \in L_t^\infty(L^{\frac{6}{5}+,1})$ and $\|\omega\|_{L^p} \approx \|\nabla u\|_{L^p}$ for $1 < p < +\infty$, then $u \in L_t^\infty(\dot{W}^{1,\frac{6}{5}+})$ and so, finally $u \in L_t^\infty(W^{1,\frac{6}{5}+}(\mathbb{R}^3))$ which is a subspace of $L_t^\infty(L^2(\mathbb{R}^3))$ with a compact embedding in the topology of $L^2_{loc}(\mathbb{R}^3)$ to a fixed t. Consequently, we can construct the solution by using only that $\omega \in L^{\frac{6}{5}} \cap L^{\frac{6}{5}+,1}$ and $\frac{\omega}{r} \in L^{\frac{6}{5}} \cap L^{\frac{6}{5}+,1}$ and by passing to the limit in a sequences of approximate axisymmetrical and regular solutions of the equation

$$\partial_t u + \mathrm{div}\,(u \otimes u) - \partial_z^2 u = -\nabla p.$$

More precisely , let $u_0 \in L^2(\mathbb{R}^3)$ such that $\omega_0 \in L^{\frac{6}{5}} \cap L^{\frac{6}{5}+,1}$ and $\frac{\omega_0}{r} \in L^{\frac{6}{5}} \cap L^{\frac{6}{5}+,1}$. Let J_n the operator which localizes in low frequencies defined by $J_n u = \mathcal{F}^{-1}(\chi(\xi 2^{-n})\mathcal{F}u(\xi))$, where $\mathcal{F}$ denotes the Fourier transform and χ is a radial and regular function, equal to which to 1 on a ball around zero. We already know that for an axisymmetrical vector field u_0, without swirl, we have that $J_n u_0$ is also axisymmetrical without swirl and also is regular (see for example [2]). So, it exists a unique regular and global in time solution, which is axisymmetrical without swirl u^n, solution of the problem

$$(NS_n)\begin{cases}\partial_t u_n + \operatorname{div}(u_n\otimes u_n) - n^{-1}\Delta_h u_n - \partial_3^2 u_n = -\nabla p_n\\ \operatorname{div} u_n = 0\\ u_n|_{t=0} = J_n u_0.\end{cases}$$

Taking into account the fact that $J_n\omega_0$ and $\frac{J_n\omega_0}{r}$ are uniformly bounded in $L^{\frac{6}{5}}\cap L^{\frac{6}{5}+,1}$ (see [8]) we obtain that u_n is a sequence which is uniformly bounded in $L^\infty_t(W^{1,\frac{6}{5}+}(\mathbb{R}^3))$. Using the equation verified by u_n we obtain easily that $\partial_t u_n$ is bounded in $L^\infty_t(H^{-N})$ for N large enough. Taking into account that the embedding of $W^{1,\frac{6}{5}+}(\mathbb{R}^3)$ in $L^2_{loc}(\mathbb{R}^3)$ is compact and as u_n is bounded in $C_{loc}(H^{-N})$ we obtain by Arzela-Ascoli lemma, up to a subsequence denoted again by u_n, that u_n converges strongly to u in $C_{loc}(H^{-N}_{loc})$. Interpolating with the fact that u_n is bounded in $L^\infty(W^{1,\frac{6}{5}+})$ we found that $u_N\to u$ in $L^\infty_{loc}(L^2(\mathbb{R}^3))$. This allows to pass to the limit in the non-linear terms and we conclude that $u_n\otimes u_n\to u\otimes u$ in $\mathcal{D}'$. Finally, by passing to the limit in the system (NS_n) we obtain a global in time, axisymmetric solution, without swirl, u of the system (NS_v).

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