

ON THE GLOBAL EXISTENCE AND UNIQUENESS OF SOLUTION TO 2-D INHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN CRITICAL SPACES

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ABSTRACT. In this paper, we establish the global existence and uniqueness of solution to 2-D inhomogeneous incompressible Navier-Stokes equations (1.1) with initial data in the critical spaces.

Precisely, under the assumption that the initial velocity u_0 in $L^2 \cap \dot{B}_{p,1}^{-1+\frac{2}{p}}$ and the initial density ρ_0 in L^∞ and having a positive lower bound, which satisfies $1 - \rho_0^{-1} \in \dot{B}_{\lambda,2}^{\frac{2}{\lambda}} \cap L^\infty$, for $p \in [2, \infty[$ and $\lambda \in [1, \infty[$ with $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$, the system (1.1) has a global solution. The solution is unique if $p = 2$. With additional assumptions on the initial density in case $p > 2$, we can also prove the uniqueness of such solution. In particular, this result improves the previous work in [2] where u_0 belongs to $\dot{B}_{2,1}^0$ and $\rho_0^{-1} - 1$ belongs to $\dot{B}_{\frac{2}{\lambda},1}^\varepsilon$, and we also remove the assumption that the initial density is close enough to a positive constant in [19] yet with additional regularities on the initial density here.

Keywords: Inhomogeneous Navier-Stokes equations, Global well-posedness, Critical spaces

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1. INTRODUCTION

In this paper, we investigate the global well-posedness of the following 2-D inhomogeneous incompressible Navier-Stokes equations:

$$(1.1) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases}$$

where the unknowns ρ and $u = (u_1, u_2)^T$ stand for the density and velocity of the fluid respectively, and Π is a scalar pressure function, which guarantees the divergence free condition of the velocity field. Such a system can be used to describe the mixture of several immiscible fluids that are incompressible and with different densities, it can also characterize a fluid containing a molten substance.

It is easy to observe that for any smooth enough solution (ρ, u) of (1.1), one has the following energy law:

$$(1.2) \quad \frac{1}{2} \int_{\mathbb{R}^2} \rho |u|^2 dx + \int_0^t \int_{\mathbb{R}^2} |\nabla u|^2 dx dt' = \frac{1}{2} \int_{\mathbb{R}^2} \rho_0 |u_0|^2 dx.$$

Based on the energy law, Kazhikov [6] proved that the d -dimensional system (1.1) (with $d = 2, 3$) has a global weak solution provided that the initial density is bounded from above and away from vacuum, the initial velocity belongs to H^1 (the size of H^1 norm should be sufficiently small in three space dimension). Danchin and Mucha [17] solved the uniqueness problem with smoother velocity. The uniqueness of Kazhikov weak solution was solved in [25] (see [11, 19, 28] for the improvements). Lately Danchin and Mucha [18] established the existence and uniqueness of such solution even allowing the appearing of vacuum. In general, DiPerna and Lions [20, 24] proved the

global existence of weak solutions to (1.1) in the energy space in any space dimensions. Yet the uniqueness and regularities of such weak solutions are listed as open questions by Lions in [24].

On the other hand, if the initial data of the density ρ is away from zero, we denote by $a \stackrel{\text{def}}{=} \rho^{-1} - 1$, then the system (1.1) can be equivalently reformulated as

$$(1.3) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u + (1 + a)(\nabla \Pi - \Delta u) = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases}$$

Just as the classical Navier-Stokes equations, which corresponds to the case when $a = 0$ in (1.3), the system (1.3) also has the following scaling-invariant property: if (a, u) solves (1.3) with initial data (a_0, u_0) , then for any $\ell > 0$,

$$(1.4) \quad (a, u)_\ell(t, x) \stackrel{\text{def}}{=} (a(\ell^2 \cdot, \ell \cdot), \ell u(\ell^2 \cdot, \ell \cdot))$$

is also a solution of (1.3) with initial data $(a_0(\ell \cdot), \ell u_0(\ell \cdot))$. We call such functional spaces as critical spaces if the norms of which are invariant under the scaling transformation $(a_0, u_0) \mapsto (a_0(\ell \cdot), \ell u_0(\ell \cdot))$.

Danchin [14] first established the global well-posedness of the system (1.3) with initial data in the almost critical Sobolev spaces. After the works [1, 4, 13] in the critical framework, Danchin and Mucha [16] eventually proved the global well-posedness of (1.3) with initial density being close enough to a positive constant in the multiplier space of $\dot{B}_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ and initial velocity being small enough in $\dot{B}_{p,1}^{-1+\frac{d}{p}}(\mathbb{R}^d)$ for $1 \leq p < 2d$. The work of [3] is the first to investigate the global well-posedness of the 3-D incompressible inhomogeneous Navier-Stokes equation with initial data in the critical spaces and yet without the size restriction on a_0 . One may check [19] and references therein for the recent progress in this direction.

In two dimensions and with initial density being bounded from above and away from vacuum, Danchin [14] proved the global well-posedness of the system (1.1) if $\rho_0^{-1} - 1 \in H^{1+\alpha}$ and $u_0 \in H^\beta$ with $\alpha, \beta > 0$. The authors of [5] proved the global existence and uniqueness of the solution to the system (1.1) with variable viscosity when the viscosity is close enough to a positive constant, and $\rho_0^{-1} - 1 \in \dot{B}_{2,1}^1 \cap \dot{B}_{\infty,\infty}^\alpha$ with $\alpha > 0$ and $u_0 \in \dot{B}_{2,1}^0$ (one may check [21] for the existence result of the system (1.1) with H^1 initial data and also [26] together with the references therein for the rough density case). Haspot [23] proved the global well-posedness of system (1.1) with small initial velocity $u_0 \in \dot{B}_{p_2,r}^{\frac{2}{p_2}-1}$ and more regular initial density $\rho_0^{-1} - 1 \in B_{p_1,\infty}^{\frac{2}{p_1}+\varepsilon}$ with some technical conditions on p_1, p_2, r and ε . Recently, the first two authors of this paper improved the above result in [2] to that $u_0 \in \dot{B}_{2,1}^0$ and $\rho_0^{-1} - 1 \in \dot{B}_{\frac{2}{\varepsilon},1}^\varepsilon$ with $M_1 \leq \rho_0 \leq M_2$ and $0 < \varepsilon < 1$. This is, to the best of our knowledge, the first global well-posedness result of (1.1) in the critical framework that does not require any smallness condition. More recently, based on interpolation results, time weighted estimates and maximal regularity estimates for time evolutionary Stokes system in Lorentz spaces (with respect to the time variable), Danchin and Wang [19] obtained the existence and uniqueness of the system (1.1) when the initial data ρ_0 is close to a positive constant in L^∞ and $u_0 \in L^2 \cap \dot{B}_{p,1}^{-1+\frac{2}{p}}$ with $1 < p < 2$.

Inspired by [2], we shall investigate the global well-posedness of the system (1.1) with initial data in the general critical spaces. The main result states as follows.

Theorem 1.1. *Let M_1, M_2 be two positive constants, $p \in [2, +\infty[$ and $\lambda \in [1, +\infty[$ with $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$. We assume that $u_0 \in L^2 \cap \dot{B}_{p,1}^{-1+\frac{2}{p}}$ is a solenoidal vector field and $1 - \rho_0^{-1} \in \dot{B}_{\lambda,2}^{\frac{2}{\lambda}} \cap L^\infty$*

satisfies

$$(1.5) \quad M_1 \leq \rho_0 \leq M_2.$$

Then the system (1.1) has a global solution $(\rho, u, \nabla\Pi)$ which satisfies

$$(1.6) \quad \begin{aligned} & \rho^{-1} - 1 \in \mathcal{C}([0, \infty[; \dot{B}_{\lambda, 2}^{\frac{2}{\lambda}} \cap L^\infty), \quad M_1 \leq \rho(t, x) \leq M_2 \quad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ & u \in \mathcal{C}([0, \infty[; L^2 \cap \dot{B}_{p, 1}^{-1+\frac{2}{p}}) \cap \tilde{L}_{loc}^1(\mathbb{R}_+; \dot{H}^2) \cap L_{loc}^1(\mathbb{R}_+; \dot{B}_{p, 1}^{1+\frac{2}{p}}), \\ & \nabla\Pi \in L_{loc}^1(\mathbb{R}_+; \dot{B}_{p, 1}^{-1+\frac{2}{p}}) \cap L_{loc}^1(\mathbb{R}_+; L^2) \quad \text{and} \quad \partial_t u \in L_{loc}^1(\mathbb{R}_+; \dot{B}_{p, 1}^{-1+\frac{2}{p}}) \cap \tilde{L}_{loc}^1(\mathbb{R}_+; L^2). \end{aligned}$$

In particular, for $p = 2$, this solution is unique. For $p \in]2, \infty[$, if in addition, $\rho_0^{-1} - 1 \in \dot{B}_{\frac{p}{p-1}, \infty}^{2-\frac{2}{p}} \cap \dot{B}_{2, 1}^1$, then the solution is unique and satisfies $\rho^{-1} - 1 \in C([0, \infty[; \dot{B}_{\frac{p}{p-1}, \infty}^{2-\frac{2}{p}} \cap \dot{B}_{2, 1}^1)$.

Notice that for $1 \leq p < 2$, $\dot{B}_{p, 1}^{-1+\frac{2}{p}} \hookrightarrow \dot{B}_{2, 1}^0$ and for $p \in [4/3, 2[$, $\lambda = \frac{2p}{2-p}$, (p, λ) satisfies $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$, then we deduce from Theorem 1.1 that the system (1.1) has a unique global-in-time solution satisfying (1.6). Precisely

Corollary 1.1. *Let $p \in [4/3, 2[$ and $u_0 \in \dot{B}_{p, 1}^{-1+\frac{2}{p}}$ be a solenoidal vector field, and ρ_0 satisfies (1.5) with $1 - \rho_0^{-1} \in \dot{B}_{\frac{2p}{2-p}, 1}^{-1+\frac{2}{p}}$. Then the system (1.1) has a unique global solution $(\rho, u, \nabla\Pi)$ which satisfies*

$$(1.7) \quad \begin{aligned} & \rho^{-1} - 1 \in \mathcal{C}([0, \infty[; \dot{B}_{\frac{2p}{2-p}, 1}^{-1+\frac{2}{p}}), \quad u \in \mathcal{C}([0, \infty[; \dot{B}_{p, 1}^{-1+\frac{2}{p}}) \cap L_{loc}^1(\mathbb{R}_+; \dot{B}_{p, 1}^{1+\frac{2}{p}}), \\ & \nabla\Pi \in L_{loc}^1(\mathbb{R}_+; \dot{B}_{p, 1}^{-1+\frac{2}{p}}), \quad \partial_t u \in L_{loc}^1(\mathbb{R}_+; \dot{B}_{p, 1}^{-1+\frac{2}{p}}). \end{aligned}$$

Remark 1.1. *In some sense, our result here removed the assumption in [19] that the initial data ρ_0 is close to a positive constant and also extends the case $p \in]1, 2[$ to $p \in [4/3, \infty[$. We believe that Corollary 1.1 is correct even for $p \in]1, 4/3[$, yet we shall not pursue this direction here.*

Remark 1.2. *The main ideas used to prove the uniqueness part of Theorem 1.1 for the cases $p = 2$ and $p \in]2, +\infty[$ are quite different. For the case when $p = 2$, we shall combine the Lagrangian approach with the techniques in [2] to deal with the difference between any two solutions of (1.1) in the L^2 framework (see Proposition 2.3 below), which is also different from the Lagrangian method in [16] where the smallness of the variation of the initial density is required. While for the case when $p \in]2, +\infty[$, without the smallness assumption on the variation of the initial density, it is difficult for us to close the estimate for the difference in the L^p framework if we use the Lagrangian approach. Instead, we shall perform the estimates in Euclidean coordinates and rely on the Osgood Lemma to conclude the uniqueness part in Section 4.*

The structure of this paper lists as follows: In Section 2, we shall first collect some basic facts on Littlewood-Paley theory, and then to apply it to study some commutator's estimates, finally we shall apply the previous estimates to investigate the linearized equations of (1.3). In Section 3, we shall derive the necessary *a priori* estimates used in the proof of Theorem 1.1. In Section 4, we shall conclude the proof of Theorem 1.1.

Notations: For two operators A, B , we denote $[A, B] = AB - BA$, the commutator between A and B . For $a \lesssim b$, we mean that there is a uniform constant C , which may be different on different lines, such that $a \leq Cb$, and C_{in} denotes a positive constant depending only on the norm to the initial data. $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$. For $r \in [1, +\infty]$ and $\bar{\mathbb{N}} \stackrel{\text{def}}{=} \mathbb{N} \cup \{-1\}$, we denote

$\{c_{q,r}\}_{q \in \mathbb{Z}}$ (or $\{c_{q,r}\}_{q \in \bar{\mathbb{N}}}$) a sequence in $\ell^r(\mathbb{Z})$ (or $\ell^r(\bar{\mathbb{N}})$) such that $\|\{c_{q,r}\}_q\|_{\ell^r} = 1$. In particular, we designate $c_{q,1}$ by d_q and $c_{q,2}$ by c_q .

For X a Banach space and I an interval of \mathbb{R} , we denote by $\mathcal{C}(I; X)$ the set of continuous functions on I with values in X , and by $\mathcal{C}_b(I; X)$ the subset of bounded functions of $\mathcal{C}(I; X)$. For $p \in [1, +\infty]$, the notation $L^p(I; X)$ stands for the set of measurable functions on I with values in X , such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(I)$.

2. PRELIMINARIES

2.1. Basic facts on Littlewood-Paley theory. The proof of Theorem 1.1 requires Littlewood-Paley theory. For the convenience of the readers, we briefly recall some basic facts in the case of $x \in \mathbb{R}^2$ (see, e.g. [7]). Let $\chi(\tau)$ and $\varphi(\tau)$ be smooth functions such that

$$\begin{aligned} \text{Supp } \varphi &\subset \left\{ \tau \in \mathbb{R} : \frac{3}{4} < \tau < \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\tau) = 1; \\ \text{Supp } \chi &\subset \left\{ \tau \in \mathbb{R} : 0 \leq \tau < \frac{4}{3} \right\} \quad \text{and} \quad \forall \tau \geq 0, \chi(\tau) + \sum_{q \geq 0} \varphi(2^{-q}\tau) = 1, \end{aligned}$$

we define the dyadic operators as follows: for $u \in \mathcal{S}'$,

$$(2.1) \quad \begin{aligned} \dot{\Delta}_q u &\stackrel{\text{def}}{=} \varphi(2^{-q}|\mathbf{D}|)u \quad \forall q \in \mathbb{Z}, & \text{and} & \quad \dot{S}_q u \stackrel{\text{def}}{=} \sum_{j \leq q-1} \dot{\Delta}_j u, \\ \Delta_q u &\stackrel{\text{def}}{=} \varphi(2^{-q}|\mathbf{D}|)u \text{ if } q \geq 0, & \Delta_{-1} u &\stackrel{\text{def}}{=} \chi(|\mathbf{D}|)u \quad \text{and} \quad S_q u \stackrel{\text{def}}{=} \sum_{j=-1}^{q-1} \Delta_j u. \end{aligned}$$

The dyadic operator satisfies the property of almost orthogonality:

$$(2.2) \quad \begin{aligned} \dot{\Delta}_k \dot{\Delta}_q u &\equiv 0 \quad \text{if } |k - q| \geq 2 \quad \text{and} \quad \dot{\Delta}_k (\dot{S}_{q-1} u \dot{\Delta}_q u) &\equiv 0 \quad \text{if } |k - q| \geq 5, \\ \Delta_k \Delta_q u &\equiv 0 \quad \text{if } |k - q| \geq 2 \quad \text{and} \quad \Delta_k (S_{q-1} u \Delta_q u) &\equiv 0 \quad \text{if } |k - q| \geq 5. \end{aligned}$$

Definition 2.1. Let $s \in \mathbb{R}$, $1 \leq p, r \leq +\infty$ and $\bar{\mathbb{N}} \stackrel{\text{def}}{=} \mathbb{N} \cup \{-1\}$, we define

(1) the inhomogeneous Besov space $B_{p,r}^s$ to be the set of distributions u in \mathcal{S}' so that

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \left\| 2^{qs} \|\Delta_q u\|_{L^p} \right\|_{\ell^r(\bar{\mathbb{N}})} < \infty,$$

(2) the homogeneous Besov space $\dot{B}_{p,r}^s$ to be the set of distributions u in \mathcal{S}'_h ($\mathcal{S}'_h \stackrel{\text{def}}{=} \{u \in \mathcal{S}', \lim_{\lambda \rightarrow +\infty} \|\theta(\lambda D)u\|_{L^\infty} = 0 \text{ for any } \theta \in \mathcal{D}(\mathbb{R}^2)\}$) so that

$$\|u\|_{\dot{B}_{p,r}^s} \stackrel{\text{def}}{=} \left\| 2^{qs} \|\dot{\Delta}_q u\|_{L^p} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

Remark 2.1. (1) We point out that if $s > 0$ then $B_{p,r}^s = \dot{B}_{p,r}^s \cap L^p$ and

$$\|u\|_{B_{p,r}^s} \approx \|u\|_{\dot{B}_{p,r}^s} + \|u\|_{L^p}.$$

(2) If $u \in B_{p,\infty}^s \cap B_{p,\infty}^{\tilde{s}}$ and $s < \tilde{s}$, $\theta \in (0, 1)$, $1 \leq p \leq \infty$, then $u \in B_{p,1}^{\theta s + (1-\theta)\tilde{s}}$ and

$$(2.3) \quad \|u\|_{B_{p,1}^{\theta s + (1-\theta)\tilde{s}}} \leq \frac{C}{\tilde{s} - s} \left(\frac{1}{\theta} + \frac{1}{1-\theta} \right) \|u\|_{B_{p,\infty}^s}^\theta \|u\|_{B_{p,\infty}^{\tilde{s}}}^{1-\theta}.$$

- (3) Let $s \in \mathbb{R}$, $1 \leq p, r \leq +\infty$, and $u \in \mathcal{S}'_h$. Then u belongs to $\dot{B}_{p,r}^s$ if and only if there exists some positive constant C and some nonnegative generic element $\{c_{q,r}\}_{q \in \mathbb{Z}}$ of $\ell^r(\mathbb{Z})$ such that $\|\{c_{q,r}\}_{q \in \mathbb{Z}}\|_{\ell^r(\mathbb{Z})} = 1$ and for any $q \in \mathbb{Z}$

$$(2.4) \quad \|\dot{\Delta}_q u\|_{L^p} \leq C c_{q,r} 2^{-qs} \|u\|_{\dot{B}_{p,r}^s}.$$

Similarly, for $u \in \mathcal{S}'$, u belongs to $B_{p,r}^s$ if and only if there holds

$$(2.5) \quad \|\Delta_q u\|_{L^p} \leq C c_{q,r} 2^{-qs} \|u\|_{B_{p,r}^s}.$$

We also recall Bernstein's inequality from [7]:

Lemma 2.1. Let $\mathcal{B} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2, |\xi| \leq \frac{4}{3}\}$ be a ball and $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ a ring. A constant C exists so that for any positive real number λ , any nonnegative integer k , any smooth homogeneous function σ of degree m , any couple of real numbers (a, b) with $b \geq a \geq 1$, and any function u in L^a , there hold

$$(2.6) \quad \begin{aligned} \text{Supp } \hat{u} \subset \lambda \mathcal{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \lambda^{k+2(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}, \\ \text{Supp } \hat{u} \subset \lambda \mathcal{C} &\Rightarrow C^{-1-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k} \lambda^k \|u\|_{L^a}, \\ \text{Supp } \hat{u} \subset \lambda \mathcal{C} &\Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m} \lambda^{m+2(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}, \end{aligned}$$

with $\sigma(D)u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\sigma \hat{u})$.

In what follows, we shall frequently use Bony's decomposition [9] in both homogeneous and inhomogeneous context. The homogeneous Bony's decomposition reads

$$(2.7) \quad uv = T_u v + T'_v u = T_u v + T_v u + R(u, v),$$

where

$$T_u v \stackrel{\text{def}}{=} \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} u \dot{\Delta}_q v, \quad T'_v u \stackrel{\text{def}}{=} \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u \dot{S}_{q+2} v, \quad R(u, v) \stackrel{\text{def}}{=} \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u \tilde{\dot{\Delta}}_q v \text{ with } \tilde{\dot{\Delta}}_q v \stackrel{\text{def}}{=} \sum_{|q'-q| \leq 1} \dot{\Delta}_{q'} v,$$

and the inhomogeneous Bony's decomposition can be defined in a similar manner.

We shall also use the following law of pr-a-product.

Proposition 2.1 (Theorems 2.47 and 2.52 in [7]). (1) There exists a constant C so that for $s \in \mathbb{R}$, $t < 0$, $p, p_1, p_2, r, r_1, r_2 \in [1, +\infty]$,

$$(2.8) \quad \begin{aligned} \|T_u v\|_{\dot{B}_{p,r}^s} &\leq C^{|s|+1} \|u\|_{L^\infty} \|v\|_{\dot{B}_{p,r}^s}, \\ \|T_u v\|_{\dot{B}_{p,r}^{s+t}} &\leq \frac{C^{|s+t|+1}}{-t} \|u\|_{\dot{B}_{p_1,r_1}^t} \|v\|_{\dot{B}_{p_2,r_2}^s} \quad \text{with} \quad \frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} \stackrel{\text{def}}{=} \min\left(1, \frac{1}{r_1} + \frac{1}{r_2}\right). \end{aligned}$$

- (2) Let (s_1, s_2) be in \mathbb{R}^2 and (p_1, p_2, r_1, r_2) be in $[1, +\infty]^4$. We assume that $\frac{1}{p} \stackrel{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\frac{1}{r} \stackrel{\text{def}}{=} \frac{1}{r_1} + \frac{1}{r_2} \leq 1$. Then there exists a constant C so that

$$\begin{aligned} \|R(u, v)\|_{\dot{B}_{p,r}^{s_1+s_2}} &\leq \frac{C^{s_1+s_2+1}}{s_1+s_2} \|u\|_{\dot{B}_{p_1,r_1}^{s_1}} \|v\|_{\dot{B}_{p_2,r_2}^{s_2}} \quad \text{if } s_1+s_2 > 0, \\ \|R(u, v)\|_{\dot{B}_{p,\infty}^0} &\leq C \|u\|_{\dot{B}_{p_1,r_1}^{s_1}} \|v\|_{\dot{B}_{p_2,r_2}^{s_2}} \quad \text{if } r = 1 \text{ and } s_1+s_2 = 0. \end{aligned}$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we shall use Chemin-Lerner type norm from [10].

Definition 2.2. Let $s \in \mathbb{R}$, $r, \lambda, p \in [1, +\infty]$ and $T > 0$. we define

$$\|u\|_{\tilde{L}_T^\lambda(B_{p,r}^s)} \stackrel{\text{def}}{=} \left\| 2^{qs} \|\Delta_q u\|_{L_T^\lambda(L^p)} \right\|_{\ell^r(\bar{\mathbb{N}})} \quad \text{and} \quad \|u\|_{\tilde{L}_T^\lambda(\dot{B}_{p,r}^s)} \stackrel{\text{def}}{=} \left\| 2^{qs} \|\dot{\Delta}_q u\|_{L_T^\lambda(L^p)} \right\|_{\ell^r(\mathbb{Z})}.$$

Finally we recall the following commutator's estimates which will be frequently used throughout this paper.

Lemma 2.2 (Lemma 1 in [27], Lemma 2.97 in [7]). (*Commutator estimate*) Let $(p, s, r) \in [1, +\infty]^3$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{s}$, θ be a C^1 function on \mathbb{R}^d such that $(1 + |\cdot|)\hat{\theta} \in L^1$. There exists a constant C such that for any function a with gradient in L^p and any function b in L^s , we have, for any positive λ ,

$$(2.9) \quad \|[\theta(\lambda^{-1}D), a]b\|_{L^r} \leq C\lambda^{-1}\|\nabla a\|_{L^p}\|b\|_{L^s}.$$

2.2. Some useful estimates. In this subsection, we shall apply the basic facts in the previous subsection to study some estimates, which will be used in the subsequent sections. We first present the following commutator's estimate, the proof of which is given for the sake of completeness.

Lemma 2.3. Let $p \in [2, \infty[$ and u be a solenoidal vector field with $\nabla u \in L^2$. Then there holds

$$(2.10) \quad \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q, u \cdot \nabla]u\|_{L^p} \lesssim \|\nabla u\|_{L^2}^2.$$

Proof. Thanks to Bony's decomposition (2.7) and the fact that $\operatorname{div} u = 0$, we decompose $[\dot{\Delta}_q, u \cdot \nabla]u$ into the following four terms:

$$(2.11) \quad [\dot{\Delta}_q, u \cdot \nabla]u = \dot{\Delta}_q(\partial_j R(u^j, u)) + \dot{\Delta}_q(T_{\partial_j u} u^j) - T'_{\dot{\Delta}_q \partial_j u} u^j + [\dot{\Delta}_q, T_{u^j}] \partial_j u \stackrel{\text{def}}{=} \sum_{i=1}^4 \mathcal{R}_q^i,$$

where repeated indices means the summation of the index from 1 to 2.

We first deduce form Lemma 2.1 that

$$\begin{aligned} \|\mathcal{R}_q^1\|_{L^p} &\lesssim 2^{q(3-\frac{2}{p})} \sum_{k \geq q-3} \|\tilde{\Delta}_k u\|_{L^2} \|\dot{\Delta}_k u^j\|_{L^2} \\ &\lesssim 2^{q(3-\frac{2}{p})} \sum_{k \geq q-3} c_k^2 2^{-2k} \|\nabla u\|_{L^2}^2 \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|\nabla u\|_{L^2}^2. \end{aligned}$$

Here and in all that follows, we always denote $\{c_q\}_{q \in \mathbb{Z}}$ to be a unit generic element of $\ell^2(\mathbb{Z})$ so that $\sum_{q \in \mathbb{Z}} c_q^2 = 1$.

While considering the support properties to the Fourier transform of the terms in $T_{\partial_j u} u^j$, we infer

$$\|\mathcal{R}_q^2\|_{L^p} \lesssim \sum_{|q-k| \leq 4} \|\dot{S}_{k-1} \nabla u\|_{L^\infty} \|\dot{\Delta}_k u\|_{L^p},$$

yet it follows from Lemma 2.1 that

$$(2.12) \quad \|\dot{S}_{k-1} \nabla u\|_{L^\infty} \lesssim \sum_{\ell \leq k-2} 2^\ell \|\dot{\Delta}_\ell \nabla u\|_{L^2} \lesssim c_k 2^k \|\nabla u\|_{L^2},$$

so that we infer

$$\begin{aligned} \|\mathcal{R}_q^2\|_{L^p} &\lesssim \sum_{|q-k| \leq 4} c_k 2^{2k(1-\frac{1}{p})} \|\nabla u\|_{L^2} \|\dot{\Delta}_k u\|_{L^2} \\ &\lesssim \sum_{|q-k| \leq 4} c_k^2 2^{k(1-\frac{2}{p})} \|\nabla u\|_{L^2}^2 \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|\nabla u\|_{L^2}^2. \end{aligned}$$

Notice that $\mathcal{R}_q^3 = -\sum_{k \geq q-3} \dot{S}_{k+2} \dot{\Delta}_q \partial_j u \dot{\Delta}_k u^j$, one has

$$\begin{aligned} \|\mathcal{R}_q^3\|_{L^p} &\lesssim \|\dot{\Delta}_q \nabla u\|_{L^\infty} \sum_{k \geq q-3} \|\dot{\Delta}_k u\|_{L^p} \\ &\lesssim 2^q \|\dot{\Delta}_q \nabla u\|_{L^2} \sum_{k \geq q-3} c_k 2^{-\frac{2}{p}k} \|\nabla u\|_{L^2} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|\nabla u\|_{L^2}^2. \end{aligned}$$

For the last term \mathcal{R}_q^4 in (2.11), we use the property of spectral localization of the Littlewood-Paley decomposition to write $\mathcal{R}_q^4 = \sum_{|k-q| \leq 4} [\dot{\Delta}_q, \dot{S}_{k-1} u^j] \dot{\Delta}_k \partial_j u$, from which, Lemma 2.2 and (2.12), we infer

$$\begin{aligned} \|\mathcal{R}_q^4\|_{L^p} &\lesssim \sum_{|k-q| \leq 4} 2^{k-q} \|\dot{S}_{k-1} \nabla u\|_{L^\infty} \|\dot{\Delta}_k u\|_{L^p} \\ &\lesssim 2^{-q} \sum_{|k-q| \leq 4} c_k^2 2^{2k(1-\frac{1}{p})} \|\nabla u\|_{L^2}^2 \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|\nabla u\|_{L^2}^2. \end{aligned}$$

By summarizing the above estimates, we arrive at (2.10), which ends the proof of Lemma 2.3. \square

Lemma 2.4. Let $p \in [2, \infty[$, $\lambda \in [1, \infty[$, $a \in \dot{B}_{\lambda,2}^{\frac{2}{\lambda}}$ and $f \in L^2$. Then there holds

$$(2.13) \quad \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q, a] f\|_{L^p} \lesssim \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}.$$

Proof. We first get, by applying Bony's decomposition (2.7), that

$$(2.14) \quad [\dot{\Delta}_q, a] f = \dot{\Delta}_q R(a, f) + \dot{\Delta}_q T_f a - T'_{\dot{\Delta}_q f} a - [\dot{\Delta}_q, T_a] f \stackrel{\text{def}}{=} \sum_{i=1}^4 \mathcal{I}_q^i.$$

In case $\lambda \leq p$, we deduce from Lemma 2.1 that

$$\begin{aligned} \|\mathcal{I}_q^1\|_{L^p} &\lesssim 2^q \sum_{k \geq q-3} \|\dot{\Delta}_k a\|_{L^p} \|\tilde{\Delta}_k f\|_{L^2} \lesssim 2^q \sum_{k \geq q-3} c_k^2 2^{-\frac{2}{p}k} \|a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \|f\|_{L^2} \\ &\lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}. \end{aligned}$$

While for $p < \lambda < \infty$, one has $\lambda > 2$ and $\frac{2\lambda}{2+\lambda} < 2 \leq p$, so that we infer

$$\begin{aligned} \|\mathcal{I}_q^1\|_{L^p} &\lesssim 2^{q(1+\frac{2}{\lambda}-\frac{2}{p})} \sum_{k \geq q-3} \|\dot{\Delta}_k a\|_{L^\lambda} \|\tilde{\Delta}_k f\|_{L^2} \\ &\lesssim 2^{q(1+\frac{2}{\lambda}-\frac{2}{p})} \sum_{k \geq q-3} c_k^2 2^{-\frac{2}{\lambda}k} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}. \end{aligned}$$

Similarly in case $\lambda \leq p$, we have

$$\begin{aligned} \|\mathcal{I}_q^3\|_{L^p} &\lesssim \sum_{k \geq q-3} \|\dot{S}_{k+2} \dot{\Delta}_q f\|_{L^\infty} \|\dot{\Delta}_k a\|_{L^p} \\ &\lesssim 2^q \|\dot{\Delta}_q f\|_{L^2} \sum_{k \geq q-3} c_k 2^{-\frac{2}{p}k} \|a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}. \end{aligned}$$

And in case $p \leq \lambda < \infty$, one has

$$\begin{aligned} \|\mathcal{I}_q^3\|_{L^p} &\lesssim \sum_{k \geq q-3} \|\dot{S}_{k+2}\dot{\Delta}_q f\|_{L^{\frac{p\lambda}{\lambda-p}}} \|\dot{\Delta}_k a\|_{L^\lambda} \\ &\lesssim c_q 2^{q(1+\frac{2}{\lambda}-\frac{2}{p})} \sum_{k \geq q-3} c_k 2^{-\frac{2}{\lambda}k} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}. \end{aligned}$$

On the other hand, we deduce from Lemma 2.1 that

$$\begin{aligned} \|\mathcal{I}_q^2\|_{L^p} &\lesssim \sum_{|q-k| \leq 4} \|\dot{S}_{k-1} f\|_{L^\infty} \|\dot{\Delta}_k a\|_{L^p} \\ &\lesssim \sum_{|q-k| \leq 4} c_k^2 2^{k(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \|f\|_{L^2} \\ &\lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \|f\|_{L^2} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}, \end{aligned}$$

in case $\lambda \leq p$. While for $p \leq \lambda < \infty$, we get, by a similar estimate of \mathcal{I}_q^3 , that

$$\begin{aligned} \|\mathcal{I}_q^2\|_{L^p} &\lesssim \sum_{|q-k| \leq 4} \|\dot{S}_{k-1} f\|_{L^{\frac{p\lambda}{\lambda-p}}} \|\dot{\Delta}_k a\|_{L^\lambda} \\ &\lesssim \sum_{|q-k| \leq 4} c_k^2 2^{k(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}, \end{aligned}$$

Finally it follows from Lemma 2.2 that

$$\begin{aligned} \|\mathcal{I}_q^4\|_{L^p} &\lesssim \sum_{|q-k| \leq 4} 2^{-q} \|\nabla \dot{S}_{k-1} a\|_{L^\infty} \|\dot{\Delta}_k f\|_{L^p} \\ &\lesssim 2^{-q} \sum_{|q-k| \leq 4} c_k^2 2^{k(2-\frac{2}{p})} \|a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \|f\|_{L^2} \\ &\lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \|f\|_{L^2} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}, \end{aligned}$$

in the case when $\lambda \leq p$. While for $p < \lambda$, we infer

$$\begin{aligned} \|\mathcal{I}_q^4\|_{L^p} &\lesssim \sum_{|q-k| \leq 4} 2^{-q} \|\nabla \dot{S}_{k-1} a\|_{L^\lambda} \|\dot{\Delta}_k f\|_{L^{\frac{p\lambda}{\lambda-p}}} \\ &\lesssim 2^{-q} \sum_{|q-k| \leq 4} c_k^2 2^{k(2-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2} \lesssim c_q^2 2^{q(1-\frac{2}{p})} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|f\|_{L^2}. \end{aligned}$$

In view of (2.14), we achieve (2.13) by summarizing the above estimates. This completes the proof of the Lemma 2.4. \square

Lemma 2.5. *Let $\lambda \in [1, \infty[$ and $p \in]2, \infty[$ with $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$. Let $g \in \dot{B}_{p,1}^{-1+\frac{2}{p}}$ and $f \in \dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}$. Then we have*

$$\begin{aligned} (2.15) \quad &\sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q \mathbb{P}, f] g\|_{L^p} \lesssim \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}}, \\ &\sum_{q \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, f] g\|_{L^2} \lesssim \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}}. \end{aligned}$$

Proof. Similar to (2.14), we write

$$(2.16) \quad [\dot{\Delta}_q \mathbb{P}, f]g = \dot{\Delta}_q \mathbb{P}R(f, g) + \dot{\Delta}_q \mathbb{P}T_g f - T'_{\dot{\Delta}_q g} f - [\dot{\Delta}_q \mathbb{P}, T_f]g \stackrel{\text{def}}{=} \sum_{i=1}^4 \mathcal{K}_q^i.$$

As $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$, we deduce from Lemma 2.1 that

$$\begin{aligned} \|\mathcal{K}_q^1\|_{L^p} &\lesssim 2^{\frac{2}{\lambda}q} \sum_{k \geq q-3} \|\dot{\Delta}_k f\|_{L^\lambda} \|\tilde{\dot{\Delta}}_k g\|_{L^p} \\ &\lesssim 2^{\frac{2}{\lambda}q} \sum_{k \geq q-3} d_k 2^{-k(\frac{2}{\lambda} + \frac{2}{p} - 1)} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \\ &\lesssim d_q 2^{q(1-\frac{2}{p})} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}. \end{aligned}$$

The same argument gives rise to

$$\|\mathcal{K}_q^1\|_{L^2} \lesssim d_q \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}.$$

Here and in all that follows, we always designate $\{d_q\}_{q \in \mathbb{Z}}$ to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{q \in \mathbb{Z}} d_q = 1$.

Thanks to the support properties of Fourier transform to the terms in $\dot{S}_{k+2} \dot{\Delta}_q g$, we get

$$\begin{aligned} \|\mathcal{K}_q^3\|_{L^p} &\lesssim \sum_{k \geq q-3} \|\dot{S}_{k+2} \dot{\Delta}_q g\|_{L^\infty} \|\dot{\Delta}_k f\|_{L^p} \lesssim 2^{\frac{2}{p}q} \|\dot{\Delta}_q g\|_{L^p} \sum_{k \geq q-3} 2^{-\frac{2}{p}k} \|f\|_{\dot{B}_{p,\infty}^{\frac{2}{p}}} \\ &\lesssim 2^{\frac{2}{p}q} d_q 2^{q(1-\frac{2}{p})} 2^{-\frac{2}{p}q} \|f\|_{\dot{B}_{p,\infty}^{\frac{2}{p}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \\ &\lesssim d_q 2^{q(1-\frac{2}{p})} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}, \end{aligned}$$

in the case when $\lambda \leq p$. While if $p \leq \lambda$, one has

$$\begin{aligned} \|\mathcal{K}_q^3\|_{L^p} &\lesssim 2^{\frac{2}{\lambda}q} \|\dot{\Delta}_q g\|_{L^p} \sum_{k \geq q-3} \|\dot{\Delta}_k f\|_{L^\lambda} \\ &\lesssim d_q 2^{q(1-\frac{2}{p})} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}. \end{aligned}$$

Similarly, one has

$$\|\mathcal{K}_q^3\|_{L^2} \lesssim d_q \|f\|_{\dot{B}_{2,\infty}^1} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim d_q \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}},$$

in the case when $\lambda \leq 2$. And for $2 \leq \lambda < \infty$, we have

$$\begin{aligned} \|\mathcal{K}_q^2\|_{L^2} &\lesssim \sum_{k \geq q-3} \|\dot{S}_{k+2} \dot{\Delta}_q g\|_{L^{\frac{2\lambda}{\lambda-2}}} \|\dot{\Delta}_k f\|_{L^\lambda} \lesssim 2^{q(\frac{2}{p} + \frac{2}{\lambda} - 1)} \|\dot{\Delta}_q g\|_{L^p} \sum_{k \geq q-3} \|\dot{\Delta}_k f\|_{L^\lambda} \\ &\lesssim 2^{q(\frac{2}{p} + \frac{2}{\lambda} - 1)} d_q 2^{q(1-\frac{2}{p})} \sum_{k \geq q-3} 2^{-\frac{2}{\lambda}k} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim d_q \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}. \end{aligned}$$

Notice that

$$\|\dot{S}_{k-1} g\|_{L^\infty} \leq \sum_{\ell \leq k-2} 2^{\frac{2}{p}\ell} \|\dot{\Delta}_\ell g\|_{L^p} \lesssim \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \sum_{\ell \leq k-2} d_\ell 2^\ell \lesssim d_k 2^k \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}},$$

so that one has

$$\begin{aligned} \|\mathcal{K}_q^2\|_{L^p} &\lesssim \sum_{|q-k| \leq 4} \|\dot{\Delta}_k f\|_{L^p} \|\dot{S}_{k-1} g\|_{L^\infty} \\ &\lesssim d_q 2^{q(1-\frac{2}{p})} \|f\|_{\dot{B}_{p,\infty}^{\frac{2}{p}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim d_q 2^{q(1-\frac{2}{p})} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}, \end{aligned}$$

in case $\lambda \leq p$. For $p \leq \lambda < \infty$, one has

$$\|\dot{S}_{k-1} g\|_{L^{\frac{p\lambda}{\lambda-p}}} \lesssim \sum_{\ell \leq k-2} 2^{\frac{2}{\lambda}\ell} \|\dot{\Delta}_\ell g\|_{L^p} \lesssim d_k 2^{k(1+\frac{2}{\lambda}-\frac{2}{p})} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}},$$

from which, we infer

$$\begin{aligned} \|\mathcal{K}_q^2\|_{L^p} &\lesssim \sum_{|q-k| \leq 4} \|\dot{\Delta}_k f\|_{L^\lambda} \|\dot{S}_{k-1} g\|_{L^{\frac{p\lambda}{\lambda-p}}} \\ &\lesssim d_q 2^{q(1-\frac{2}{p})} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}. \end{aligned}$$

Similarly, one has

$$\|\mathcal{K}_q^2\|_{L^2} \lesssim d_q \|f\|_{\dot{B}_{2,\infty}^1} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim d_q \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}},$$

in case $\lambda \leq 2$. And for $2 \leq \lambda < \infty$, we have

$$\|\dot{S}_{k-1} g\|_{L^{\frac{2\lambda}{\lambda-2}}} \lesssim \sum_{\ell \leq k-2} 2^{(\frac{2}{\lambda}+\frac{2}{p}-1)\ell} \|\dot{\Delta}_\ell g\|_{L^p} \lesssim d_k 2^{\frac{2}{\lambda}k} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}},$$

from which, we infer

$$\begin{aligned} \|\mathcal{K}_q^2\|_{L^2} &\lesssim \sum_{|q-k| \leq 4} \|\dot{\Delta}_k f\|_{L^\lambda} \|\dot{S}_{k-1} g\|_{L^{\frac{2\lambda}{\lambda-2}}} \\ &\lesssim d_q \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}. \end{aligned}$$

Observing that

$$\|\nabla \dot{S}_{k-1} f\|_{L^\infty} \lesssim \sum_{\ell \leq k-2} 2^{k(1+\frac{2}{\lambda})} \|\dot{\Delta}_\ell f\|_{L^\lambda} \lesssim 2^k \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}},$$

so that we deduce from Lemma 2.2 that

$$\begin{aligned} \|\mathcal{K}_q^4\|_{L^p} &\lesssim 2^{-q} \sum_{|q-k| \leq 4} \|\nabla \dot{S}_{k-1} f\|_{L^\infty} \|\dot{\Delta}_k g\|_{L^p} \\ &\lesssim d_q 2^{q(1-\frac{2}{p})} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}. \end{aligned}$$

While due to $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda}$ and $p < \infty$, we get, by using the inequality (2.9), that

$$\begin{aligned} \|\mathcal{K}_q^4\|_{L^2} &\lesssim 2^{-q} \sum_{|q-k| \leq 4} \|\nabla \dot{S}_{k-1} f\|_{L^{\frac{2p}{p-2}}} \|\dot{\Delta}_k g\|_{L^p} \\ &\lesssim \sum_{|q-k| \leq 4} 2^{-q} 2^{\frac{2}{p}k} \|\dot{\Delta}_k g\|_{L^p} \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \\ &\lesssim d_q \|f\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|g\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}. \end{aligned}$$

By summarizing the above estimates, we arrive at (2.15). This completes the proof of Lemma 2.5. \square

Lemma 2.6. Let $p \in [2, \infty[$ and $\lambda \in [1, \infty[$. Let $u \in \dot{B}_{p,1}^{1+\frac{2}{p}} \cap H^1$ and $a \in \dot{B}_{\lambda,2}^{\frac{2}{\lambda}}$. Then we have

$$(2.17) \quad \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q \mathbb{P}, \dot{S}_m a] \Delta u\|_{L^p} \lesssim 2^m \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \left(\|\nabla u\|_{L^2} + \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}}^{\frac{1}{2}} \right).$$

Proof. Once again similar to (2.14), we write

$$(2.18) \quad \begin{aligned} [\dot{\Delta}_q \mathbb{P}, \dot{S}_m a] \Delta u &= \dot{\Delta}_q \mathbb{P} R(\dot{S}_m a, \Delta u) + \dot{\Delta}_q \mathbb{P} T_{\Delta u} \dot{S}_m a \\ &\quad - T'_{\dot{\Delta}_q \Delta u} \dot{S}_m a - [\dot{\Delta}_q \mathbb{P}, T_{\dot{S}_m a}] \Delta u \stackrel{\text{def}}{=} \sum_{i=1}^4 \mathcal{L}_q^i. \end{aligned}$$

It follows from Lemma 2.1 that

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|\mathcal{L}_q^1\|_{L^p} &\lesssim \sum_{q \in \mathbb{Z}} 2^q \sum_{k \geq q-3} 2^{-k} \|\dot{\Delta}_k \nabla^2 \dot{S}_m a\|_{L^2} \|\dot{\Delta}_k \nabla u\|_{L^2} \\ &\lesssim \sum_{q \in \mathbb{Z}} 2^q \sum_{k \geq q-3} c_k^2 2^{-k} \|\nabla^2 \dot{S}_m a\|_{L^2} \|\nabla u\|_{L^2} \\ &\lesssim 2^m \|\nabla a\|_{L^2} \|\nabla u\|_{L^2} \lesssim 2^m \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla u\|_{L^2}, \end{aligned}$$

in case $\lambda \leq 2$. If $2 \leq \lambda < \infty$, we have

$$\sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|\mathcal{L}_q^1\|_{L^p} \leq \sum_{q \in \mathbb{Z}} 2^{q\frac{2}{\lambda}} \|\dot{\Delta}_q \mathbb{P} R(\dot{S}_m a, \Delta u)\|_{L^{\frac{2\lambda}{\lambda+2}}} \lesssim 2^m \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla u\|_{L^2}.$$

Notice that $\|\dot{S}_{k-1} \Delta u\|_{L^\infty} \lesssim c_k 2^{2k} \|\nabla u\|_{L^2}$, we infer

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|\mathcal{L}_q^2\|_{L^p} &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \sum_{|q-k| \leq 4} \|\dot{S}_{k-1} \Delta u\|_{L^\infty} \|\dot{\Delta}_k \dot{S}_m a\|_{L^p} \\ &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \sum_{|q-k| \leq 4} c_k^2 2^{k(1-\frac{2}{p})} \|\nabla \dot{S}_m a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \|\nabla u\|_{L^2} \\ &\lesssim 2^m \|a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \|\nabla u\|_{L^2} \lesssim 2^m \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla u\|_{L^2}, \end{aligned}$$

in case $\lambda \leq p$. When $p \leq \lambda$, we have

$$\|\dot{S}_{k-1} \Delta u\|_{L^{\frac{p\lambda}{\lambda-p}}} \lesssim \sum_{\ell \leq k-2} 2^{2\ell(1+\frac{1}{\lambda}-\frac{1}{p})} \|\dot{\Delta}_\ell \nabla u\|_{L^2} \lesssim c_k 2^{2k(1+\frac{1}{\lambda}-\frac{1}{p})} \|\nabla u\|_{L^2},$$

so that one has

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|\mathcal{L}_q^2\|_{L^p} &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \sum_{|q-k| \leq 4} \|\dot{S}_{k-1} \Delta u\|_{L^{\frac{p\lambda}{\lambda-p}}} \|\dot{\Delta}_k \dot{S}_m a\|_{L^\lambda} \\ &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \sum_{|q-k| \leq 4} c_k^2 2^{k(1-\frac{2}{p})} \|\nabla \dot{S}_m a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla u\|_{L^2} \\ &\lesssim 2^m \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla u\|_{L^2}. \end{aligned}$$

Considering the support properties to the Fourier transform of the terms in $\dot{S}_{k+2}\dot{\Delta}_q\Delta u$, we deduce

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|\mathcal{L}_q^3\|_{L^p} &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \sum_{k \geq q-3} \|\dot{\Delta}_q \Delta u\|_{L^\infty} \|\dot{\Delta}_k \dot{S}_m a\|_{L^p} \\ &\lesssim \sum_{q \in \mathbb{Z}} c_q 2^{q(1+\frac{2}{p})} \|\nabla u\|_{L^2} \sum_{k \geq q-3} \|\dot{\Delta}_k \dot{S}_m a\|_{L^p} \\ &\lesssim 2^m \|\nabla u\|_{L^2} \|a\|_{\dot{B}_{p,2}^{\frac{2}{p}}} \lesssim 2^m \|\nabla u\|_{L^2} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}}, \end{aligned}$$

in case $\lambda \leq p$. If $p \leq \lambda$, we have

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|\mathcal{L}_q^3\|_{L^p} &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \sum_{k \geq q-3} \|\dot{\Delta}_q \Delta u\|_{L^{\frac{p\lambda}{\lambda-p}}} \|\dot{\Delta}_k \dot{S}_m a\|_{L^\lambda} \\ &\lesssim \sum_{q \in \mathbb{Z}} \sum_{k \geq q-3} 2^{(q-k)(1+\frac{2}{\lambda})} \|\dot{\Delta}_q \nabla u\|_{L^2} 2^{k\frac{2}{\lambda}} \|\dot{\Delta}_k \nabla \dot{S}_m a\|_{L^\lambda} \\ &\lesssim 2^m \|\nabla u\|_{L^2} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}}. \end{aligned}$$

Finally we deduce from Lemma 2.2 that

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|\mathcal{L}_q^4\|_{L^p} &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(-2+\frac{2}{p})} \sum_{|k-q| \leq 4} \|\nabla \dot{S}_{k-1} \dot{S}_m a\|_{L^\infty} 2^{2k} \|\dot{\Delta}_k u\|_{L^p} \\ &\lesssim \sum_{q \in \mathbb{Z}} \sum_{|k-q| \leq 4} 2^{(-2+\frac{2}{p})(q-k)} 2^{k\frac{2}{p}} \|\dot{\Delta}_k \nabla u\|_{L^p} \|\nabla \dot{S}_m a\|_{\dot{B}_{\lambda,1}^{\frac{2}{\lambda}}} \\ &\lesssim 2^m \|a\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}}. \end{aligned}$$

While for any N we observe that

$$\|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \lesssim \sum_{q \leq N} 2^q \|\dot{\Delta}_q u\|_{L^2} + \sum_{q \geq N} 2^{q(\frac{2}{p}+1)} 2^{-q} \|\dot{\Delta}_q u\|_{L^p} \lesssim 2^N \|u\|_{L^2} + 2^{-N} \|u\|_{\dot{B}_{p,\infty}^{1+\frac{2}{p}}}.$$

By taking $N \in \mathbb{Z}$ in the above inequality so that

$$2^{2N} \approx \|u\|_{L^2}^{-1} \|u\|_{\dot{B}_{p,\infty}^{1+\frac{2}{p}}},$$

we obtain

$$(2.19) \quad \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \lesssim \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}}^{\frac{1}{2}},$$

As a result, it comes out

$$\sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|\mathcal{L}_q^4\|_{L^p} \lesssim 2^m \|a\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}}^{\frac{1}{2}}.$$

By summarizing the above estimates, we arrive at (2.17), which completes the proof of this lemma. \square

By taking the space divergence to the momentum equation of (1.3), we find

$$(2.20) \quad \operatorname{div}((1+a)\nabla\Pi) = \operatorname{div}(a\Delta u) - \operatorname{div}(u \cdot \nabla u).$$

Let us start with the estimate of $\|\nabla\Pi\|_{L^2}$.

Lemma 2.7. Let $p \in [2, +\infty]$ and $\lambda \in [1, +\infty[$ satisfy $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$. Let $a \in \dot{B}_{\lambda,2}^{\frac{2}{\lambda}} \cap L^\infty$, $\nabla \Pi \in L^2$, and $u \in \dot{H}^1$ with $\operatorname{div} u = 0$ satisfy the equation (2.20). Then under the assumption that $1 + a \geq c$, for any $m \in \mathbb{Z}^+$, there holds

$$(2.21) \quad \|\nabla \Pi\|_{L^2} \lesssim \|a - \dot{S}_m a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}} + 2^m \|\nabla u\|_{L^2} \|a\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}} \cap L^\infty} + \|\nabla u\|_{L^2}^2.$$

Proof. In view of (2.20), for any $m \in \mathbb{Z}^+$, we write

$$\operatorname{div}((1+a)\nabla \Pi) = \operatorname{div}((a - \dot{S}_m a)\Delta u) + \operatorname{div}(\dot{S}_m a \Delta u) - \operatorname{div}(u \cdot \nabla u).$$

By applying Bony's decomposition (2.7) and $\operatorname{div} u = 0$, we obtain

$$(2.22) \quad \begin{aligned} \operatorname{div}((1+a)\nabla \Pi) &= \operatorname{div} T_{\Delta u}(a - \dot{S}_m a) + \operatorname{div} R(\Delta u, a - \dot{S}_m a) + T_{\nabla(a - \dot{S}_m a)} \Delta u \\ &\quad + \operatorname{div}(\dot{S}_m a \Delta u) - \operatorname{div}(u \cdot \nabla u). \end{aligned}$$

Then we get, by taking L^2 inner product of (2.22) with Π and using $1 + a \geq c$, that

$$(2.23) \quad \begin{aligned} c \|\nabla \Pi\|_{L^2}^2 &\leq \|\nabla \Pi\|_{L^2} \left(\|T_{\Delta u}(a - \dot{S}_m a)\|_{L^2} + \|R(\Delta u, a - \dot{S}_m a)\|_{L^2} \right. \\ &\quad \left. + \|T_{\nabla(a - \dot{S}_m a)} \Delta u\|_{\dot{H}^{-1}} + \|\operatorname{div}(u \cdot \nabla) u\|_{\dot{H}^{-1}} + \|\operatorname{div}(\dot{S}_m a \Delta u)\|_{\dot{H}^{-1}} \right). \end{aligned}$$

It follows from the law of product in Besov spaces, Proposition 2.1, and $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$ that

$$\begin{aligned} \|T_{\Delta u}(a - \dot{S}_m a)\|_{L^2} &\lesssim \|\Delta u\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \|a - \dot{S}_m a\|_{\dot{B}_{\frac{2p}{p-2},2}^{1-\frac{2}{p}}} \lesssim \|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}} \|a - \dot{S}_m a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}}, \\ \|R(\Delta u, a - \dot{S}_m a)\|_{L^2} &\lesssim \|R(\Delta u, a - \dot{S}_m a)\|_{\dot{B}_{\frac{p\lambda}{p+\lambda},2}^{\frac{2}{p}+\frac{2}{\lambda}-1}} \lesssim \|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}} \|a - \dot{S}_m a\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}}, \\ \|T_{\nabla(a - \dot{S}_m a)} \Delta u\|_{\dot{H}^{-1}} &\lesssim \|\Delta u\|_{\dot{B}_{p,2}^{-1+\frac{2}{p}}} \|\nabla(a - \dot{S}_m a)\|_{\dot{B}_{\frac{2p}{p-2},\infty}^{-\frac{2}{p}}} \lesssim \|u\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}} \|a - \dot{S}_m a\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}}. \end{aligned}$$

While again by using Bony's decomposition (2.7) and $\operatorname{div} u = 0$, we find

$$\begin{aligned} \|\operatorname{div}(\dot{S}_m a \Delta u)\|_{\dot{H}^{-1}} &\lesssim \|T_{\nabla \dot{S}_m a} \Delta u\|_{\dot{H}^{-1}} + \|T_{\Delta u} \nabla \dot{S}_m a\|_{\dot{H}^{-1}} + \|R(\dot{S}_m a, \Delta u)\|_{L^2} \\ &\lesssim \|\nabla \dot{S}_m a\|_{L^\infty} \|\nabla u\|_{L^2} + \|\dot{S}_m a\|_{\dot{B}_{\lambda,\infty}^{1+\frac{2}{\lambda}}} \|\Delta u\|_{\dot{B}_{p,2}^{\frac{2}{p}-2}} \\ &\lesssim 2^m \|\nabla u\|_{L^2} (\|a\|_{L^\infty} + \|a\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}}). \end{aligned}$$

Finally again due to $\operatorname{div} u = 0$, we have $\operatorname{div}(u \cdot \nabla u) = \sum_{i,j=1}^2 \partial_i u_j \partial_j u_i$. As a result, it comes out

$$\|\operatorname{div}(u \cdot \nabla u)\|_{\dot{H}^{-1}} \lesssim \|T_{\partial u} \partial u\|_{\dot{H}^{-1}} + \|R(u, \nabla u)\|_{L^2} \lesssim \|\nabla u\|_{L^2}^2.$$

By substituting the above estimates into (2.23), we obtain (2.21). This completes the proof of Lemma 2.7. \square

2.3. The estimate of the linearized equation. The goal of this subsection is to present some *a priori* estimates to the linearized equations of (1.1). We start with the following estimates concerning the solution of transport equation.

Proposition 2.2. Let $m \in \mathbb{Z}$ and $\lambda \in [1, \infty]$, let the convection velocity u satisfy $\nabla u \in L_T^1(L^\infty) \cap \tilde{L}_T^1(\dot{H}^2)$ and $\operatorname{div} u = 0$. Then for $a_0 \in \dot{B}_{\lambda,2}^{\frac{2}{\lambda}}$, the following equation

$$(2.24) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ a|_{t=0} = a_0. \end{cases}$$

has a unique solution $a \in C([0, T]; \dot{B}_{\lambda,2}^{\frac{2}{\lambda}})$ so that for all $t \in (0, T]$,

$$(2.25) \quad \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \lesssim \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} (1 + \|u\|_{\tilde{L}_t^1(\dot{H}^2)}) e^{C\|\nabla u\|_{L_t^1(L^\infty)}},$$

and

$$(2.26) \quad \begin{aligned} \|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} &\lesssim \left(\sum_{q \geq m} 2^{\frac{4}{\lambda}q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2 \right)^{\frac{1}{2}} \\ &+ \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} (e^{C\|\nabla u\|_{L_t^1(L^\infty)}} - 1) + \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|u\|_{\tilde{L}_t^1(\dot{H}^2)} e^{C\|\nabla u\|_{L_t^1(L^\infty)}}. \end{aligned}$$

Proof. The proof of the above proposition is similar to that of Proposition 2.3 in [2], for completeness, we just outline it here. We divide the proof into the following three cases: $\lambda > 2$, $1 < \lambda \leq 2$, and $\lambda = 1$.

Case 1: $\lambda > 2$. By applying the dyadic operator $\dot{\Delta}_q$ to the transport equation of (2.24), we write

$$\partial_t \dot{\Delta}_q a + u \cdot \nabla \dot{\Delta}_q a = -[\dot{\Delta}_q, u \cdot \nabla] a,$$

from which and $\operatorname{div} u = 0$, we infer

$$(2.27) \quad \|\dot{\Delta}_q a\|_{L_t^\infty(L^\lambda)} \leq \|\dot{\Delta}_q a_0\|_{L^\lambda} + \int_0^t \|[\dot{\Delta}_q, u \cdot \nabla] a\|_{L^\lambda} d\tau.$$

It follows from classical commutator's estimate (see Lemma 2.100 and Remark 2.102 in [7]) that

$$(2.28) \quad \|[\dot{\Delta}_q, u \cdot \nabla] a(t)\|_{L^\lambda} \lesssim c_q(t) 2^{-\frac{2}{\lambda}q} \|\nabla u\|_{L^\infty} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}},$$

which along with (2.27) ensures that

$$(2.29) \quad \|\dot{\Delta}_q a\|_{L_t^\infty(L^\lambda)} \leq \|\dot{\Delta}_q a_0\|_{L^\lambda} + C 2^{-\frac{2}{\lambda}q} \int_0^t c_q(\tau) \|\nabla u\|_{L^\infty} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} d\tau.$$

Hence, one has

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \leq \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} + C \int_0^t \|\nabla u\|_{L^\infty} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} d\tau.$$

from which, we get, by applying Gronwall's inequality, that

$$(2.30) \quad \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \leq \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} e^{C\|\nabla u\|_{L_t^1(L^\infty)}}.$$

On the other hand, considering $q \geq m$ in (2.29), we infer

$$(2.31) \quad \|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \leq \left(\sum_{q \geq m} 2^{\frac{4}{\lambda}q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2 \right)^{\frac{1}{2}} + C \int_0^t \|\nabla u\|_{L^\infty} \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} d\tau.$$

By plugging (2.30) into (2.31), we obtain

$$\|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \leq \left(\sum_{q \geq m} 2^{\frac{4}{\lambda}q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2 \right)^{\frac{1}{2}} + \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} (e^{C\|\nabla u\|_{L_t^1(L^\infty)}} - 1).$$

Case 2: $1 < \lambda \leq 2$. It is easy to observe that $\partial_j a$ satisfies

$$(2.32) \quad \partial_t \partial_j a + u \cdot \nabla \partial_j a = -\partial_j u \cdot \nabla a.$$

Applying the operator $\dot{\Delta}_q$ to (2.32) gives

$$\partial_t \dot{\Delta}_q \partial_j a + u \cdot \nabla \partial_j \dot{\Delta}_q a = -\dot{\Delta}_q (\partial_j u \cdot \nabla a) - [\dot{\Delta}_q, u \cdot \nabla] \partial_j a,$$

from which and $\operatorname{div} u = 0$, we infer

$$\|\dot{\Delta}_q \partial_j a\|_{L^\lambda} \leq \|\dot{\Delta}_q \partial_j a_0\|_{L^\lambda} + \int_0^t \|\dot{\Delta}_q (\partial_j u \cdot \nabla a)\|_{L^\lambda} d\tau + \int_0^t \|[\dot{\Delta}_q, u \cdot \nabla] \partial_j a\|_{L^\lambda} d\tau.$$

Since $0 \leq \frac{2}{\lambda} - 1 < 1$, we deduce from classical commutator's estimate (see Lemma 2.100 and Remark 2.102 in [7]) that

$$(2.33) \quad \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \leq \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} + \int_0^t \|\partial_j u \cdot \nabla a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}-1}} d\tau + C \int_0^t \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla u\|_{L^\infty} d\tau.$$

Notice that

$$\begin{aligned} \|\partial_j u \cdot \nabla a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}-1}} &\lesssim \|T_{\partial_j u} \nabla a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}-1}} + \|R(\partial_j u, a)\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} + \|T_{\nabla a} \partial_j u\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}-1}} \\ &\lesssim \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla u\|_{L^\infty} + \|u\|_{\dot{H}^2} \|\nabla a\|_{\dot{B}_{\frac{2\lambda}{2-\lambda},\infty}^{-2+\frac{2}{\lambda}}}, \end{aligned}$$

and

$$\|\nabla a\|_{L_t^\infty(\dot{B}_{\frac{2\lambda}{2-\lambda},\infty}^{-2+\frac{2}{\lambda}})} \lesssim \|\nabla a\|_{L_t^\infty(\dot{B}_{2,\infty}^0)} \lesssim \|\nabla a_0\|_{\dot{B}_{2,\infty}^0} e^{C\|\nabla u\|_{L_t^1(L^\infty)}} \lesssim \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} e^{C\|\nabla u\|_{L_t^1(L^\infty)}},$$

we infer

$$\|\partial_j u \cdot \nabla a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}-1}} \leq \|a\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla u\|_{L^\infty} + C\|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|u\|_{\dot{H}^2} e^{C\|\nabla u\|_{L_t^1(L^\infty)}}.$$

By inserting the above estimate into (2.33) and then applying Gronwall's inequality, we obtain (2.25). Exactly along the same line, we deduce (2.26).

Case 3: $\lambda = 1$. Taking one more space derivative the equation (2.32) gives

$$\partial_t \partial_{ij}^2 a + u \cdot \nabla \partial_{ij}^2 a = -\partial_i u \cdot \nabla \partial_j a - \partial_j u \cdot \nabla \partial_i a - \partial_{ij}^2 u \cdot \nabla a,$$

from which, we deduce from classical commutator's estimate (see Lemma 2.100 and Remark 2.102 in [7]), Bony's decomposition and $\operatorname{div} u = 0$, that

$$\begin{aligned} \|\partial_{ij}^2 a\|_{\tilde{L}_t^\infty(\dot{B}_{1,2}^0)} &\leq \|\partial_{ij}^2 a_0\|_{\dot{B}_{1,2}^0} + C \int_0^t \|\nabla u\|_{L^\infty} \|a\|_{\dot{B}_{1,2}^2} d\tau + \|T_{\partial \nabla a} \partial u\|_{\tilde{L}_t^\infty(\dot{B}_{1,2}^0)} + \|T_{\nabla a} \partial^2 u\|_{\tilde{L}_t^1(\dot{B}_{1,2}^0)} \\ &\leq \|\partial_{ij}^2 a_0\|_{\dot{B}_{1,2}^0} + C \int_0^t (\|\nabla u\|_{L^\infty} \|a\|_{\dot{B}_{1,2}^2} + \|\nabla a\|_{L^2} \|u\|_{\dot{H}^2}) d\tau. \end{aligned}$$

Yet notice that

$$\|\nabla a\|_{\tilde{L}_t^\infty(L^2)} \lesssim \|a_0\|_{\dot{H}^1} e^{C\|\nabla u\|_{L_t^1(L^\infty)}} \lesssim \|a_0\|_{\dot{B}_{1,2}^2} e^{C\|\nabla u\|_{L_t^1(L^\infty)}},$$

we find

$$\|\partial_{ij}^2 a\|_{\tilde{L}_t^\infty(\dot{B}_{1,2}^0)} \leq \|\partial_{ij}^2 a_0\|_{\dot{B}_{1,2}^0} + C \int_0^t (\|\nabla u\|_{L^\infty} \|a\|_{\dot{B}_{1,2}^2} + \|a_0\|_{\dot{B}_{1,2}^2} \|u\|_{\dot{H}^2} e^{C\|\nabla u\|_{L_t^1(L^\infty)}}) d\tau.$$

Applying Gronwall's inequality leads to (2.25). Similar argument yields (2.26). This completes the proof of Proposition 2.2. \square

In order to prove the uniqueness part of Theorem 1.1 for $p = 2$, it is necessary for us to study the following Stokes system:

$$(2.34) \quad \left\{ \begin{array}{l} \rho_0 \partial_t u - \Delta u + \nabla \Pi = f, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \nabla \cdot u = \operatorname{div} g, \\ \Delta \Phi = \operatorname{div} g, \\ u|_{t=0} = u_0. \end{array} \right.$$

Proposition 2.3. Let $p \in [2, +\infty[$ and $\lambda \in [1, +\infty[$ satisfy $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$. We assume that $1+a_0 \stackrel{\text{def}}{=} \frac{1}{\rho_0} \geq \frac{1}{M} > 0$ for some positive constant M , $u \in \tilde{L}_T^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}}) \cap L_T^1(\dot{B}_{p,1}^{1+\frac{2}{p}})$, and $\nabla\Pi \in L_T^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})$ solves the system (2.34) for smooth enough f and g . Then there exists a large enough integer m_0 , which depends only on $\|a_0\|_{\dot{B}_{\lambda,2}^{-\frac{2}{\lambda}}}$, so that for any $m \geq m_0$, one has

$$(2.35) \quad \begin{aligned} \|u\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|(u_t, \nabla^2 u, \nabla\Pi)\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} &\lesssim \|\nabla\Phi\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}}) \cap L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ &+ (2^m \sqrt{t} + 2^{2m} t) \left(\|(u_0, \nabla\Phi_0)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2} + \|(f, \nabla \operatorname{div} g, \nabla\Phi_t)\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2)} \right). \end{aligned}$$

Proof. Let $v \stackrel{\text{def}}{=} u - \nabla\Phi$. Then in view of (2.34), one has

$$(2.36) \quad \begin{cases} \partial_t v - (1+a_0)(\Delta v - \nabla\Pi) = K, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \nabla \cdot v = 0, \\ v|_{t=0} = v_0, \end{cases}$$

where $K \stackrel{\text{def}}{=} (1+a_0)f - \partial_t \nabla\Phi + (1+a_0)\nabla \operatorname{div} g$ and $1+a_0 \stackrel{\text{def}}{=} \frac{1}{\rho_0}$.

We first get, by using L^2 energy estimate to the equation (2.34), that

$$(2.37) \quad \|\sqrt{\rho_0}v\|_{L_t^\infty(L^2)} + \|\nabla v\|_{L_t^2(L^2)} \lesssim \|\sqrt{\rho_0}v_0\|_{L^2} + \|(\sqrt{\rho_0}\partial_t \nabla\Phi, f, \nabla \operatorname{div} g)\|_{L_t^1(L^2)}.$$

In what follows, we separate the proof of (2.35) into the following two steps:

Step 1. The estimate of $\|v\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|v\|_{L_T^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}$.

By applying the operator $\mathbb{P}\dot{\Delta}_q$ to (2.36) and using a commutator process, we write

$$\partial_t \dot{\Delta}_q v - (1+a_0)\Delta \dot{\Delta}_q v = [\mathbb{P}\dot{\Delta}_q, a_0](\Delta v - \nabla\Pi) + \mathbb{P}\dot{\Delta}_q K,$$

which can be equivalently written as

$$\rho_0 \partial_t \dot{\Delta}_q v - \Delta \dot{\Delta}_q v = \rho_0 [\mathbb{P}\dot{\Delta}_q, a_0](\Delta v - \nabla\Pi) + \rho_0 \mathbb{P}\dot{\Delta}_q K,$$

By taking L^2 product of the above equation with $|\dot{\Delta}_q v|^{p-2} \dot{\Delta}_q v$ and using integration by parts, we obtain

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \|\rho_0^{\frac{1}{p}} \dot{\Delta}_q v\|_{L^p}^p - \int_{\mathbb{R}^2} \nabla \cdot (\nabla \dot{\Delta}_q v) \cdot |\dot{\Delta}_q v|^{p-2} \dot{\Delta}_q v \, dx \\ &\leq M \|\dot{\Delta}_q v\|_{L^p}^{p-1} (\|[\mathbb{P}\dot{\Delta}_q, a_0](\Delta v - \nabla\Pi)\|_{L^p} + \|\mathbb{P}\dot{\Delta}_q K\|_{L^p}). \end{aligned}$$

It follows from Lemma 8 in appendix of [15] that

$$-\int_{\mathbb{R}^2} \nabla \cdot (\nabla \dot{\Delta}_q v) \cdot |\dot{\Delta}_q v|^{p-2} \dot{\Delta}_q v \, dx \geq c_p 2^{2q} \|\dot{\Delta}_q v\|_{L^p}^p,$$

so that we have

$$(2.38) \quad \begin{aligned} &\frac{d}{dt} \|\rho_0^{\frac{1}{p}} \dot{\Delta}_q v\|_{L^p}^p + p c_p M^{-1} 2^{2q} \|\rho_0^{\frac{1}{p}} \dot{\Delta}_j v\|_{L^p}^p \\ &\lesssim \|\rho_0^{\frac{1}{p}} \dot{\Delta}_j v\|_{L^p}^{p-1} (\|[\mathbb{P}\dot{\Delta}_q, a_0](\Delta v - \nabla\Pi)\|_{L^p} + \|\mathbb{P}\dot{\Delta}_q K\|_{L^p}), \end{aligned}$$

which implies

$$\begin{aligned} \|\dot{\Delta}_q v(t)\|_{L^p} &\lesssim e^{-c_p 2^{2q} t} \|\dot{\Delta}_q v_0\|_{L^p} \\ &+ \int_0^t e^{-c_p 2^{2q}(t-\tau)} (\|[\mathbb{P}\dot{\Delta}_q, a_0](\Delta v - \nabla\Pi)\|_{L^p} + \|\mathbb{P}\dot{\Delta}_q K\|_{L^p}) \, d\tau. \end{aligned}$$

Then for any $r \in [1, +\infty]$, by taking L_t^r norm to the above inequality, we achieve

$$\|\dot{\Delta}_q v\|_{L_t^r(L^p)} \lesssim 2^{-\frac{2}{r}q} \|\dot{\Delta}_q v_0\|_{L^p} + 2^{-\frac{2}{r}q} (\|[\mathbb{P}\dot{\Delta}_q, a_0](\Delta v - \nabla \Pi)\|_{L_t^1(L^p)} + \|\mathbb{P}\dot{\Delta}_q K\|_{L_t^1(L^p)}).$$

By multiplying the above inequality by $2^{q(-1+\frac{2}{p}+\frac{2}{r})}$ and taking the ℓ^1 norm of the resulting inequalities, we find

$$(2.39) \quad \begin{aligned} \|v\|_{\tilde{L}_t^r(\dot{B}_{p,1}^{-1+\frac{2}{p}+\frac{2}{r}})} &\lesssim \|v_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \|K\|_{\tilde{L}_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \\ &+ \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (\|[\mathbb{P}\dot{\Delta}_q, a_0]\nabla \Pi\|_{L_t^1(L^p)} + \|[\mathbb{P}\dot{\Delta}_q, a_0]\Delta v\|_{L_t^1(L^p)}). \end{aligned}$$

It follows from Lemma 2.4 that

$$\sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} \|[\mathbb{P}\dot{\Delta}_q, a_0]\nabla \Pi\|_{L_t^1(L^p)} \lesssim \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla \Pi\|_{L_t^1(L^2)}.$$

While we get, by using Lemmas 2.5-2.6, that

$$\begin{aligned} \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} \|[\mathbb{P}\dot{\Delta}_q, a_0]\Delta v\|_{L_t^1(L^p)} &\lesssim \|a_0 - \dot{S}_m a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|v\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ &+ 2^m \sqrt{t} \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} (\|\nabla v\|_{L_t^2(L^2)} + \|v\|_{L_t^\infty(L^2)}^{\frac{1}{2}} \|v\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}^{\frac{1}{2}}). \end{aligned}$$

Finally we deduce from Lemma 2.7 that

$$\|\nabla \Pi\|_{L^2} \lesssim \|a_0 - \dot{S}_m a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|v\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}} + 2^m \|\nabla v\|_{L^2} \|a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}} \cap L^\infty} + \|K\|_{L^2}.$$

As a consequence, we obtain

$$\begin{aligned} \|v\|_{\tilde{L}_t^r(\dot{B}_{p,1}^{\frac{2}{p}-1+\frac{2}{r}})} &\lesssim \|v_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + (1 + \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}}) (\|K\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2)} + \|a_0 - \dot{S}_m a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|v\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}) \\ &+ 2^m \sqrt{t} \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}} \cap L^\infty} (\|\nabla v\|_{L_t^2(L^2)} + \|v\|_{L_t^\infty(L^2)}^{\frac{1}{2}} \|v\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}^{\frac{1}{2}}). \end{aligned}$$

By taking $r = \infty$ and $r = 1$ and using (2.37), we deduce that for $m \geq m_0$ with m_0 being large enough

$$(2.40) \quad \begin{aligned} \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|v\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} &\lesssim \|v_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \|K\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2)} \\ &+ (2^m \sqrt{t} + 2^{2m} t) (\|v_0\|_{L^2} + \|\nabla \Phi_t\|_{L_t^1(L^2)} + \|f\|_{L_t^1(L^2)} + \|\nabla \operatorname{div} g\|_{L_t^1(L^2)}). \end{aligned}$$

Step 2. The estimate of $\|\nabla \Pi\|_{L_T^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|v_t\|_{L_T^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}$.

By applying the operator $\operatorname{div} \dot{\Delta}_q$ to the equation (2.36), and using a standard commutator's argument, we find

$$(2.41) \quad \operatorname{div}((1+a_0)\nabla \dot{\Delta}_q \Pi) = -\dot{\Delta}_q \operatorname{div} K + \dot{\Delta}_q \operatorname{div}(a_0 \Delta v) + \operatorname{div}[\dot{\Delta}_q, a_0] \nabla \Pi.$$

By taking L^2 inner product of (2.41) with $|\dot{\Delta}_q \Pi|^{p-2} \dot{\Delta}_q \Pi$ and using $\operatorname{div} u = 0$, we find

$$\begin{aligned} 2^{2q} \|\dot{\Delta}_q \Pi\|_{L^p}^p &\lesssim - \int_{\mathbb{R}^3} \operatorname{div}((1+a_0)\dot{\Delta}_q \nabla \Pi) |\dot{\Delta}_q \Pi|^{p-2} \dot{\Delta}_q \Pi \, dx \\ &\lesssim 2^q \|\dot{\Delta}_q \Pi\|_{L^p}^{p-1} (\|\dot{\Delta}_q K\|_{L^p} + \|\dot{\Delta}_q(a_0 \Delta v)\|_{L^p} + \|[\dot{\Delta}_q, a_0] \nabla \Pi\|_{L^p}), \end{aligned}$$

from which, we infer

$$(2.42) \quad \|\nabla \Pi\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim \|K\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \|a_0 \Delta v\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q, a_0] \nabla \Pi\|_{L^p}.$$

It follows from Proposition 2.1 that

$$\|T_{a_0} \Delta v\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim \|a_0\|_{L^\infty} \|v\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}},$$

and

$$\|T_{\Delta v} a_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim \begin{cases} \|a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|\Delta v\|_{\dot{B}_{\infty,1}^{-1}}, & \text{if } \lambda \leq p, \\ \|a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|\Delta v\|_{\dot{B}_{\frac{p\lambda}{\lambda-p},1}^{-1-\frac{2}{\lambda}+\frac{2}{p}}}, & \text{if } p \leq \lambda. \end{cases}$$

Whereas as $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$, one has

$$\|T_{\Delta v} a_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim \|a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|v\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}},$$

and

$$\|R(a_0, \Delta v)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim \|R(a_0, \Delta v)\|_{\dot{B}_{\frac{p\lambda}{p+\lambda},1}^{-1+\frac{2}{p}+\frac{2}{\lambda}}} \lesssim \|a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}} \|v\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}},$$

which yields

$$\|a_0 \Delta v\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim (\|a_0\|_{L^\infty} + \|a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}}}) \|v\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}}.$$

While applying Lemma 2.4 gives rise to

$$\sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q, a_0] \nabla \Pi\|_{L^p} \lesssim \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \|\nabla \Pi\|_{L^2}.$$

Consequently, we obtain

$$\|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \lesssim \|K\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|v\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla \Pi\|_{L_t^1(L^2)}.$$

On the other hand, we deduce from the momentum equation in (2.36) that

$$\|v_t\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \lesssim (1 + \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}} \cap L^\infty}) (\|v\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}) + \|K\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}.$$

Therefore, we obtain

$$(2.43) \quad \begin{aligned} & \|v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|(\nabla^2 v, \nabla \Pi, v_t)\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \lesssim \|v_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \\ & + \|K\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2)} + (2^m \sqrt{t} + 2^{2m} t) (\|v_0\|_{L^2} + \|\partial_t \nabla \Phi\|_{L_t^1(L^2)} + \|f\|_{L_t^1(L^2)}). \end{aligned}$$

It follows from the law of product that

$$\begin{aligned} \|K\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2)} & \lesssim (1 + \|a_0\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}} \cap L^\infty}) (\|f\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2)} + \|\nabla \operatorname{div} g\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2)} \\ & + \|\nabla \Phi_t\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2)}). \end{aligned}$$

By inserting the above estimate into (2.43), we achieve (2.35). This completes the proof of Proposition 2.3. \square

To prove the uniqueness part of Theorem 1.1 for $p \in]2, \infty[$, we recall the following propositions from [2].

Proposition 2.4. (See [2]) Let $u_0 \in B_{2,\infty}^{-1}$ and $v \in L_T^1(B_{\infty,1}^1)$ be a solenoidal vector field. Let $f \in \tilde{L}_T^1(B_{2,\infty}^{-1})$, and $a \in \tilde{L}_T^\infty(\dot{B}_{2,1}^2)$ with $1+a \geq \frac{1}{M} > 0$ for some positive constant M . We assume that $u \in L_T^\infty(B_{2,\infty}^{-1}) \cap \tilde{L}_T^1(B_{2,\infty}^1)$ and $\nabla\Pi \in \tilde{L}_T^1(B_{2,\infty}^{-1})$ solves

$$(2.44) \quad \begin{cases} \partial_t u + v \cdot \nabla u - (1+a)(\Delta u - \nabla\Pi) = f, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

Then there holds:

$$\|u\|_{L_T^\infty(B_{2,\infty}^{-1})} + \|u\|_{\tilde{L}_T^1(B_{2,\infty}^1)} \leq C e^{C(T+T\|\nabla a\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^1)}^2 + \|v\|_{L_T^1(B_{\infty,1}^1)})} \\ \times (\|u_0\|_{B_{2,\infty}^{-1}} + \|f\|_{\tilde{L}_T^1(B_{2,\infty}^{-1})} + \|a\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^2)} \|\nabla\Pi\|_{\tilde{L}_T^1(B_{2,\infty}^{-1})}).$$

Proposition 2.5. (See [2]) We assume that $a \in \dot{B}_{2,1}^1$ satisfies $0 < \underline{b} \leq 1+a \leq \bar{b}$ for two positive constants \underline{b} and \bar{b} , and

$$(2.45) \quad \|a - \dot{S}_k a\|_{\dot{B}_{2,1}^1} \leq c$$

for some sufficiently small positive constant c and some integer $k \in \mathbb{N}$. Let $F \in B_{2,\infty}^{-1}$ and $\nabla\Pi \stackrel{\text{def}}{=} \mathcal{H}_b(F) \in B_{2,\infty}^{-1}$ solve

$$\operatorname{div}((1+a)\nabla\Pi) = \operatorname{div} F.$$

Then there holds

$$\|\nabla\Pi\|_{B_{2,\infty}^{-1}} \lesssim (1 + 2^k \|a\|_{B_{\infty,1}^0} (1 + \|a\|_{B_{\infty,1}^0})) (\|F\|_{B_{2,\infty}^{-2}} + \|\operatorname{div} F\|_{B_{2,\infty}^{-2}}).$$

3. A prior ESTIMATES

The goal of this section is to present the *a priori* estimates which will be the most crucial ingredient used to prove Theorem 1.1. The main result states as follows.

Proposition 3.1. Let $p \in [2, +\infty[$ and $\lambda \in [1, +\infty[$ such that $\frac{1}{2} < \frac{1}{p} + \frac{1}{\lambda} \leq 1$. Let $a_0 \in L^\infty \cap \dot{B}_{\lambda,2}^{\frac{2}{\lambda}}$ with $\rho_0 = \frac{1}{1+a_0}$ satisfying (1.5) and $u_0 \in \dot{B}_{p,1}^{-1+\frac{2}{p}} \cap L^2$. Let $(a, u, \nabla\Pi)$ be a smooth enough solution of (1.3). Then there are a positive constant C , a large integer $m_0 \in \mathbb{N}$ and a small positive time T_1 so that for $m \geq m_0$,

$$(3.1) \quad \begin{aligned} & \|\nabla u\|_{L_{T_1}^2(L^2)} + \|u\|_{\tilde{L}_{T_1}^1(\dot{H}^2)} + \|u\|_{L_{T_1}^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla\Pi\|_{L_{T_1}^1(L^2)} \\ & \leq C \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (1 - e^{-c_p 2^{2q} T_1}) \|\dot{\Delta}_q u_0\|_{L^p} + C \left(\sum_{q \in \mathbb{Z}} (1 - e^{-c 2^{2q} T_1}) \|\dot{\Delta}_q u_0\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \quad + C \left(2^{2m} T_1 + \left(\sum_{q \geq m} 2^{\frac{4}{\lambda} q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

Proof. We divide the proof into the following three steps.

Step 1. The estimate of $\|\nabla u\|_{L_t^2(L^2)}$.

We first observe that there holds (1.2) and it follows from (1.5) that

$$(3.2) \quad M_1 \leq \rho(t, x) \leq M_2.$$

While similar to the proof of Proposition 2.3, we get, by first dividing the momentum equation of (1.1) by ρ and then applying Leray projector \mathbb{P} to the resulting equation, that

$$\partial_t u + \mathbb{P}(u \cdot \nabla u) - \mathbb{P}(\rho^{-1}(\Delta u - \nabla\Pi)) = 0.$$

By applying $\dot{\Delta}_q$ to the above equation and using a standard commutator's process, we write

$$\rho \partial_t \dot{\Delta}_q u + \rho u \cdot \nabla \dot{\Delta}_q u - \Delta \dot{\Delta}_j u = -\rho [\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u + \rho [\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi).$$

By taking L^2 inner product of the above equation with $\dot{\Delta}_q u$ and using the transport equation of (1.1), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \rho |\dot{\Delta}_q u|^2 dx - \int_{\mathbb{R}^2} \Delta \dot{\Delta}_q u \cdot \dot{\Delta}_q u dx \\ & \leq \|\dot{\Delta}_q u\|_{L^2} (\|\rho [\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L^2} + \|\rho [\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L^2}). \end{aligned}$$

Observing from Lemma 2.1 that

$$-\int_{\mathbb{R}^2} \Delta \dot{\Delta}_q u \cdot \dot{\Delta}_q u dx = \int_{\mathbb{R}^2} |\nabla \dot{\Delta}_j u|^2 dx \geq \bar{c} 2^{2q} \|\dot{\Delta}_q u\|_{L^2}^2.$$

we deduce that for $c = \bar{c}/M$,

$$\begin{aligned} (3.3) \quad & \frac{d}{dt} \|\sqrt{\rho} \dot{\Delta}_q u\|_{L^2}^2 + 2c 2^{2q} \|\sqrt{\rho} \dot{\Delta}_q u\|_{L^2}^2 \\ & \lesssim \|\sqrt{\rho} \dot{\Delta}_q u\|_{L^2} (\|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L^2} + \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L^2}), \end{aligned}$$

from which, we infer

$$\begin{aligned} (3.4) \quad & \|\sqrt{\rho} \dot{\Delta}_q u(t)\|_{L^2} \lesssim e^{-c 2^{2q} t} \|\sqrt{\rho_0} \dot{\Delta}_q u_0\|_{L^2} \\ & + \int_0^t e^{-c 2^{2q}(t-t')} (\|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L^2} + \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L^2})(\tau) d\tau. \end{aligned}$$

As a consequence, we deduce from Definition 2.2 that

$$\begin{aligned} (3.5) \quad & \|\nabla u\|_{L_t^2(L^2)} \lesssim \left(\sum_{q \in \mathbb{Z}} (1 - e^{-c 2^{2q} t}) \|\dot{\Delta}_q u_0\|_{L^2}^2 \right)^{\frac{1}{2}} + \left(\sum_{j \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}} \\ & + \left(\sum_{j \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

In what follows, we shall handle term by term above. We first get, by applying (2.10) with $p = 2$, that

$$\left(\sum_{q \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}} \lesssim \|\nabla u\|_{L_t^2(L^2)}^2.$$

While it follows from (2.13) that

$$\left(\sum_{q \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] \nabla \Pi\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}} \leq \int_0^t \sum_{q \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] \nabla \Pi\|_{L^2} d\tau \lesssim \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \|\nabla \Pi\|_{L_t^1(L^2)},$$

which together with Lemma 2.7 ensures that

$$\begin{aligned} \left(\sum_{q \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] \nabla \Pi\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}} & \lesssim \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \left(\|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \right. \\ & \left. + 2^m \sqrt{t} (\|a\|_{L_t^\infty(L^\infty)} + \|a\|_{L_t^\infty(\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}})}) \|\nabla u\|_{L_t^2(L^2)} + \|\nabla u\|_{L_t^2(L^2)}^2 \right). \end{aligned}$$

On the other hand, notice that

$$(3.6) \quad [\dot{\Delta}_q \mathbb{P}, \rho^{-1}] \Delta u = [\dot{\Delta}_q \mathbb{P}, a] \Delta u = [\dot{\Delta}_q \mathbb{P}, a - \dot{S}_m a] \Delta u + [\dot{\Delta}_q \mathbb{P}, \dot{S}_m a] \Delta u.$$

we deduce from Lemma 2.5 that

$$\left(\sum_{q \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, a - \dot{S}_m a] \Delta u\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}} \lesssim \|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}.$$

Whereas it follows from Lemma 2.6 that

$$(3.7) \quad \left(\sum_{q \in \mathbb{Z}} \|[\dot{\Delta}_q \mathbb{P}, \dot{S}_m a] \Delta u\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}} \lesssim 2^m \sqrt{t} \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} (\|\nabla u\|_{L_t^2(L^2)} + \|u\|_{L_t^\infty(L^2)}^{\frac{1}{2}} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}^{\frac{1}{2}}).$$

By substituting the above estimates into (3.5), we obtain that for $2 \leq p < \infty$,

$$\begin{aligned} \|\nabla u\|_{L_t^2(L^2)} &\leq \left(\sum_{q \in \mathbb{Z}} (1 - e^{-c2^{2q}t}) \|\dot{\Delta}_q u_0\|_{L^2}^2 \right)^{\frac{1}{2}} + (1 + \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})}) \left(\|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \right. \\ &\quad \left. + 2^m \sqrt{t} \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} (\|\nabla u\|_{L_t^2(L^2)} + \|u\|_{L_t^\infty(L^2)}^{\frac{1}{2}} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}^{\frac{1}{2}}) + \|\nabla u\|_{L_t^2(L^2)}^2 \right). \end{aligned}$$

By inserting the estimates (2.25) and (2.26) into the above inequality, for any $\eta > 0$, we find

$$(3.8) \quad \begin{aligned} \|\nabla u\|_{L_t^2(L^2)} &\leq C \left(\sum_{q \in \mathbb{Z}} (1 - e^{-c2^{2q}t}) \|\dot{\Delta}_q u_0\|_{L^2}^2 \right)^{\frac{1}{2}} + \eta \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ &\quad + C_\eta (1 + \|u\|_{\tilde{L}_t^1(\dot{H}^2)}) e^{C\|\nabla u\|_{L_t^1(L^\infty)}} \left((2^{2m}t + \|\nabla u\|_{L_t^2(L^2)}^2) \right. \\ &\quad \left. + \left(\left(\sum_{q \geq m} 2^{\frac{4}{\lambda}q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2 \right)^{\frac{1}{2}} + (e^{C\|\nabla u\|_{L_t^1(L^\infty)}} - 1) \right) \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \right). \end{aligned}$$

Step 2. The estimate of $\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}$.

We first get, by a similar derivation of (2.38), that

$$\frac{d}{dt} \|\rho^{\frac{1}{p}} \dot{\Delta}_q u\|_{L^p}^p + p c_p 2^{2j} \|\dot{\Delta}_q u\|_{L^p}^p \leq \|\dot{\Delta}_q u\|_{L^p}^{p-1} (\|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L^p} + \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L^p}).$$

from which and a similar derivation of (3.4), we infer

$$(3.9) \quad \begin{aligned} \|\dot{\Delta}_q u(t)\|_{L^p} &\lesssim e^{-c_p 2^{2j} t} \|\dot{\Delta}_q u_0\|_{L^p} + \int_0^t e^{-c_p 2^{2j}(t-\tau)} \|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L^p} d\tau \\ &\quad + \int_0^t e^{-c_p 2^{2j}(t-\tau)} \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L^p} d\tau. \end{aligned}$$

Then we deduce from Definition 2.2 that

$$(3.10) \quad \begin{aligned} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (1 - e^{-c_p 2^{2q}t}) \|\dot{\Delta}_q u_0\|_{L^p} \\ &\quad + \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} \left(\|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L_t^1(L^p)} + \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L_t^1(L^p)} \right). \end{aligned}$$

It follows from (2.10) that

$$\sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L_t^1(L^p)} \lesssim \|\nabla u\|_{L_t^2(L^2)}^2.$$

Whereas we get, by applying Lemmas 2.7-2.4, that

$$\sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] \nabla \Pi\|_{L_t^1(L^p)} \lesssim \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \|\nabla \Pi\|_{L_t^1(L^2)},$$

and

$$(3.11) \quad \begin{aligned} \|\nabla \Pi\|_{L_t^1(L^2)} &\lesssim \|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla u\|_{L_t^2(L^2)}^2 \\ &+ 2^m \sqrt{t} (\|a\|_{\tilde{L}_t^\infty(L^\infty)} + \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}})}) \|\nabla u\|_{L_t^2(L^2)}. \end{aligned}$$

And it follows from Lemmas 2.5-2.6 that

$$\sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q \mathbb{P}, a - \dot{S}_m a] \Delta u\|_{L_t^1(L^p)} \lesssim \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \|a - \dot{S}_m a\|_{L_t^\infty(\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}})}$$

and

$$\sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q \mathbb{P}, \dot{S}_m a] \Delta u\|_{L_t^1(L^p)} \lesssim 2^m \sqrt{t} \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} (\|\nabla u\|_{L_t^2(L^2)} + \|u\|_{L_t^\infty(L^2)}^{\frac{1}{2}} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}^{\frac{1}{2}}).$$

By substituting the above estimates into (3.10), we achieve

$$(3.12) \quad \begin{aligned} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} &\leq C \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (1 - e^{-c_p 2^{2j} t}) \|\dot{\Delta}_q u_0\|_{L^p} + \eta \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ &+ C_\eta (1 + \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})}) \left(\|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + 2^{2m} t \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}})}^2 + \|\nabla u\|_{L_t^2(L^2)}^2 \right). \end{aligned}$$

By taking η to be small enough and substituting the estimates (2.25) and (2.26) into the resulting inequality, we arrive at

$$(3.13) \quad \begin{aligned} \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} &\lesssim \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (1 - e^{-c_p 2^{2j} t}) \|\dot{\Delta}_q u_0\|_{L^p} \\ &+ (1 + \|u\|_{\tilde{L}_t^1(\dot{H}^2)}) e^{C\|\nabla u\|_{L_t^1(L^\infty)}} ((2^{2m} t + \|\nabla u\|_{L_t^2(L^2)}^2) \\ &+ ((\sum_{q \geq m} 2^{\frac{4}{\lambda}q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2)^{\frac{1}{2}} + (e^{C\|\nabla u\|_{L_t^1(L^\infty)}} - 1)) \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}). \end{aligned}$$

Step 3. The closing of the estimate.

In view of (3.4), we get, by using a similar derivation of (3.8), that

$$(3.14) \quad \begin{aligned} \|u\|_{\tilde{L}_t^1(\dot{H}^2)} &\leq C \left(\sum_{q \in \mathbb{Z}} (1 - e^{-c_p 2^{2q} t}) \|\dot{\Delta}_q u_0\|_{L^2}^2 \right)^{\frac{1}{2}} + \eta \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ &+ C_\eta (1 + \|u\|_{\tilde{L}_t^1(\dot{H}^2)}) e^{C\|\nabla u\|_{L_t^1(L^\infty)}} (2^{2m} t + \|\nabla u\|_{L_t^2(L^2)}^2) \\ &+ ((\sum_{q \geq m} 2^{\frac{4}{\lambda}q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2)^{\frac{1}{2}} + (e^{C\|\nabla u\|_{L_t^1(L^\infty)}} - 1)) \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})}. \end{aligned}$$

Let us denote

$$U(t) \stackrel{\text{def}}{=} \|\nabla u\|_{L_t^2(L^2)} + \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|u\|_{\tilde{L}_t^1(\dot{H}^2)}.$$

Then by summarizing the estimates (3.8), (3.13) and (3.14), we achieve

$$(3.15) \quad \begin{aligned} U(t) &\leq C \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (1 - e^{-c_p 2^{2q} t}) \|\dot{\Delta}_q u_0\|_{L^p} + C \left(\sum_{q \in \mathbb{Z}} (1 - e^{-c_p 2^{2q} t}) \|\dot{\Delta}_q u_0\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &+ C_\eta e^{CU(t)} (1 + U(t)) \left(2^{2m} t + U^2(t) + ((\sum_{q \geq m} 2^{\frac{4}{\lambda}q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2)^{\frac{1}{2}} + (e^{CU(t)} - 1)) U(t) \right). \end{aligned}$$

Then by first taking $m \geq m_0$ with m_0 being large enough, and then T_1 being sufficiently small in (3.15), we deduce that there exists a sufficiently small constant C_0 so that

$$(3.16) \quad \|\nabla u\|_{L^2_{T_1}(L^2)} + \|u\|_{\tilde{L}^1_{T_1}(\dot{H}^2)} + \|u\|_{L^1_{T_1}(\dot{B}_{p,1}^{1+\frac{2}{p}})} \leq C_0,$$

which along with (3.15) ensures

$$(3.17) \quad \begin{aligned} & \|\nabla u\|_{L^2_{T_1}(L^2)} + \|u\|_{\tilde{L}^1_{T_1}(\dot{H}^2)} + \|u\|_{L^1_{T_1}(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ & \leq C \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (1 - e^{-c_p 2^{2q} T_1}) \|\dot{\Delta}_q u_0\|_{L^p} + C \left(\sum_{q \in \mathbb{Z}} (1 - e^{-c 2^{2q} T_1}) \|\dot{\Delta}_q u_0\|_{L^2}^2 \right)^{\frac{1}{2}} \\ & \quad + C \left(2^{2m} T_1 + \left(\sum_{q \geq m} 2^{\frac{4}{\lambda} q} \|\dot{\Delta}_q a_0\|_{L^\lambda}^2 \right)^{\frac{1}{2}} \right). \end{aligned}$$

By summarizing the estimates (3.11), (2.25), (2.26) and (3.17), we obtain (3.1) and

$$(3.18) \quad \|\nabla u\|_{L^2_{T_1}(L^2)} + \|u\|_{\tilde{L}^1_{T_1}(\dot{H}^2)} + \|u\|_{L^1_{T_1}(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla \Pi\|_{L^1_{T_1}(L^2)} \leq C C_0,$$

which completes the proof of Proposition 3.1. \square

Proposition 3.2. *Under the assumptions of Proposition 3.1, for any $t \in [0, T_1]$ with T_1 being determined by Proposition 3.1, we have*

$$(3.19) \quad \|u\|_{\tilde{L}^\infty_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|\partial_t u\|_{\tilde{L}^1_t(L^2)} \leq C_{\text{in}} \quad \text{and} \quad \|\nabla \Pi\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|\partial_t u\|_{L^1_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \leq C_{\text{in}},$$

where the constant C_{in} depends only on the initial data (ρ_0, u_0) .

Proof. We first deduce from (3.9) and Definition 2.2, that

$$\|u\|_{\tilde{L}^\infty_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \leq \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (\|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L^1_t(L^p)} + \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L^1_t(L^p)}),$$

from which, we get, by a similar derivation of (3.13), that

$$\begin{aligned} \|u\|_{\tilde{L}^\infty_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} & \lesssim \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \|u\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ & \quad + (1 + \|u\|_{\tilde{L}^1_t(\dot{H}^2)}) e^{C \|\nabla u\|_{L^1_t(L^\infty)}} \left((2^{2m} t + \|\nabla u\|_{L^2_t(L^2)}^2) \right. \\ & \quad \left. + \left(\left(\sum_{q \geq m} 2^{\frac{4}{\lambda} q} \|\dot{\Delta}_q \nabla a_0\|_{L^\lambda}^2 \right)^{\frac{1}{2}} + (e^{C \|\nabla u\|_{L^1_t(L^\infty)}} - 1) \right) \|u\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{2}{p}})} \right), \end{aligned}$$

which together with (1.2) and (3.18) ensures that

$$(3.20) \quad \|u\|_{\tilde{L}^\infty_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \leq C_{\text{in}}.$$

While in view of the momentum equations in (1.1), (1.2) and (3.1), we infer

$$\|\partial_t u\|_{\tilde{L}^1_t(L^2)} \lesssim (1 + \|a\|_{L^\infty_t(L^\infty)}) (\|\Delta u\|_{\tilde{L}^1_t(L^2)} + \|\nabla \Pi\|_{L^1_t(L^2)} + \|u\|_{L^2_t(L^\infty)} \|\nabla u\|_{L^2_t(L^2)}) \leq C_{\text{in}},$$

where we used the fact that

$$\|u\|_{L^2_t(L^\infty)} \leq \|u\|_{L^2_t(\dot{B}_{p,1}^{\frac{2}{p}})} \lesssim \|u\|_{\tilde{L}^\infty_t(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|u\|_{L^1_t(\dot{B}_{p,1}^{1+\frac{2}{p}})}.$$

This leads to the first inequality of (3.19).

To estimate the pressure function Π , we get, by applying the operator div to the momentum equation of (1.3) and using $\text{div } u = 0$, that

$$\text{div}[(1 + a) \nabla \Pi] = -\text{div}[(u \cdot \nabla) u] + \text{div}(a \Delta u),$$

from which, we infer

$$\operatorname{div}[(1+a)\nabla\dot{\Delta}_q\Pi] = -\dot{\Delta}_q\operatorname{div}[(u\cdot\nabla)u] + \dot{\Delta}_q\operatorname{div}(a\Delta u) + \operatorname{div}[\dot{\Delta}_q, a]\nabla\Pi.$$

By taking L^2 inner product of the above equation with $|\dot{\Delta}_q\Pi|^{p-2}\dot{\Delta}_q\Pi$ and using $\operatorname{div}u = 0$, we deduce from (1.5) that for $p > 1$

$$\begin{aligned} 2^{2q}\|\dot{\Delta}_q\Pi\|_{L^p}^p &\lesssim -\int_{\mathbb{R}^3} \operatorname{div}((1+a)\dot{\Delta}_q\nabla\Pi)|\dot{\Delta}_q\Pi|^{p-2}\dot{\Delta}_q\Pi dx \\ &\lesssim 2^q\|\dot{\Delta}_q\Pi\|_{L^p}^{p-1}(2^q\|\dot{\Delta}_q(u\otimes u)\|_{L^p} + \|\dot{\Delta}_q(a\Delta u)\|_{L^p} + \|[\dot{\Delta}_q, a]\nabla\Pi\|_{L^p}), \end{aligned}$$

which implies

$$(3.21) \quad \|\nabla\Pi\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} \lesssim \|u\otimes u\|_{\dot{B}_{p,1}^{\frac{2}{p}}} + \|a\Delta u\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \sum_{q\in\mathbb{Z}} 2^{q(-1+\frac{2}{p})}\|[\dot{\Delta}_q, a]\nabla\Pi\|_{L^p}.$$

Since $\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$ is an Banach algebra for $p < +\infty$, one has

$$\|u\otimes u\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \lesssim \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}}^2.$$

While it follows from the law of product in Besov space and Lemma 2.4 that

$$\begin{aligned} \|a\Delta u\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \sum_{q\in\mathbb{Z}} 2^{q(-1+\frac{2}{p})}\|[\dot{\Delta}_q, a]\nabla\Pi\|_{L^p} &\leq \|a\|_{L_t^\infty(L^\infty\cap\dot{B}_{\lambda,\infty}^{\frac{2}{p}})}\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ &\quad + \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{p}})}\|\nabla\Pi\|_{L_t^1(L^2)}. \end{aligned}$$

By substituting the above estimates and (3.18) into (3.21), we deduce from (2.25) that

$$\begin{aligned} (3.22) \quad \|\nabla\Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} &\lesssim \|u\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})}^2 + \|a\|_{L_t^\infty(L^\infty\cap\dot{B}_{\lambda,\infty}^{\frac{2}{p}})}\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{p}})}\|\nabla\Pi\|_{L_t^1(L^2)} \\ &\lesssim \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})}\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \\ &\quad + \|a\|_{L_t^\infty(\dot{B}_{\lambda,2}^{\frac{2}{p}}\cap L^\infty)}(\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla\Pi\|_{L_t^1(L^2)}) \lesssim C_{\text{in}}. \end{aligned}$$

Finally we deduce from the momentum equation of (1.3) and the law of product in Besov spaces that

$$(3.23) \quad \|\partial_t u\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \lesssim \|u\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})}^2 + (1 + \|a\|_{L_t^\infty(L^\infty\cap\dot{B}_{\lambda,\infty}^{\frac{2}{p}})})(\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla\Pi\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})}) \lesssim C_{\text{in}}.$$

which together with (3.22) ensures the second inequality of (3.19). This completes the proof of Proposition 3.2. \square

4. THE PROOF OF THEOREM 1.1

In this section, we present the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof of Theorem 1.1 into two steps.

Step 1. Existence of strong solutions.

We first mollify the initial data to be

$$(4.1) \quad a_{0,n} \stackrel{\text{def}}{=} a_0 * j_n, \quad \text{and} \quad u_{0,n} \stackrel{\text{def}}{=} u_0 * j_n,$$

where $j_n(|x|) = n^2 j(|x|/n)$ is the standard Friedrich's mollifier. Then we deduce from the standard well-posedness theory of inhomogeneous Navier-Stokes system (see [14, 25] for instance) that (1.1) has a unique global solution $(\rho_n, u_n, \nabla \Pi_n)$. It is easy to observe from (4.1) that

$$\|a_{0,n}\|_{L^\infty \cap \dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \leq C \|a_0\|_{L^\infty \cap \dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} \quad \text{and} \quad \|u_{0,n}\|_{L^2 \cap \dot{B}_{p,1}^{-1+\frac{2}{p}}} \leq C \|u_0\|_{L^2 \cap \dot{B}_{p,1}^{-1+\frac{2}{p}}}.$$

Then under the assumptions of Theorem 1.1, we deduce from the proof of Propositions 3.1 and 3.2 that there exists a positive time T_1 so that

$$(4.2) \quad \begin{aligned} & \|u_n\|_{\tilde{L}_{T_1}^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}}) \cap L_{T_1}^1(\dot{B}_{p,1}^{1+\frac{2}{p}}) \cap \tilde{L}_{T_1}^1(\dot{H}^2)} + \|\partial_t u_n\|_{\tilde{L}_{T_1}^1(L^2) \cap L_{T_1}^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|\nabla \Pi_n\|_{L_{T_1}^1(L^2) \cap L_{T_1}^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} \\ & + \|a_n\|_{\tilde{L}_{T_1}^\infty(\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}) \cap L_{T_1}^\infty(L^\infty)} \leq C_{\text{in}} \quad \text{and} \quad M_1 \leq \rho_n(t, x) \leq M_2. \end{aligned}$$

With (4.2), we get, by using a standard compactness argument (see [13] for instance), that (1.1) has a solution $(\rho, u, \nabla \Pi)$ on $[0, T_1]$ so that

$$(4.3) \quad \begin{aligned} & a \in C([0, T_1]; \dot{B}_{\lambda,2}^{\frac{2}{\lambda}} \cap L^\infty), \quad u \in C([0, T_1]; \dot{B}_{p,1}^{-1+\frac{2}{p}}) \cap L_{T_1}^1(\dot{B}_{p,1}^{1+\frac{2}{p}}) \cap \tilde{L}_{T_1}^1(\dot{H}^2), \\ & \partial_t u \in \tilde{L}_{T_1}^1(L^2) \cap L_{T_1}^1(\dot{B}_{p,1}^{-1+\frac{2}{p}}), \quad \nabla \Pi \in L_{T_1}^1(L^2) \cap L_{T_1}^1(\dot{B}_{p,1}^{-1+\frac{2}{p}}) \quad \text{and} \quad M_1 \leq \rho(t, x) \leq M_2. \end{aligned}$$

Then there exists $t_0 \in (0, T_1)$ such that $u(t_0) \in H^1$, and it follows from (4.3) that $a(t_0) \in \dot{B}_{\lambda,2}^{\frac{2}{\lambda}} \cap L^\infty$ and $u(t_0) \in \dot{B}_{p,1}^{-1+\frac{2}{p}}$. As a consequence, with initial data at time t_0 , (1.1) has a global solution (see [17, 25] for instance) $(\rho, u, \nabla \Pi)$ so that

$$(4.4) \quad \begin{aligned} & (\partial_t u, \nabla^2 u, \nabla \Pi) \in (L^2([t_0, +\infty[; L^2))^3, \quad u \in L^\infty([t_0, +\infty[; H^1), \\ & \nabla u \in L^2([t_0, +\infty[; L^2) \cap L_{\text{loc}}^1([t_0, \infty[; L^\infty)). \end{aligned}$$

On the other hand, we deduce from the inequality (3.9) that for $t \geq t_0$

$$\begin{aligned} & \|u\|_{\tilde{L}^\infty([t_0, t); \dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|u\|_{L^1([t_0, t); \dot{B}_{p,1}^{1+\frac{2}{p}})} \lesssim \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (1 - e^{-c_p 2^{2j} t}) \|\dot{\Delta}_q u(t_0)\|_{L^p} \\ & + \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} (\|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L^1([t_0, t); L^p)} + \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L^1([t_0, t); L^p)}), \end{aligned}$$

from which, Lemmas 2.4 and 2.3 and (4.4), we deduce that

$$\begin{aligned} & \|u\|_{\tilde{L}^\infty([t_0, t); \dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|u\|_{L^1([t_0, t); \dot{B}_{p,1}^{1+\frac{2}{p}})} \lesssim \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \|\nabla u\|_{L^2([t_0, t); L^2)}^2 \\ & + \|a\|_{\tilde{L}^\infty([t_0, t); \dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \|\Delta u - \nabla \Pi\|_{L^1([t_0, t); L^2)} \leq C(t), \end{aligned}$$

which along with Proposition 2.2 ensures that

$$\|a\|_{\tilde{L}^\infty([t_0, t); \dot{B}_{\lambda,2}^{\frac{2}{\lambda}})} \lesssim \|a(t_0)\|_{\dot{B}_{\lambda,2}^{\frac{2}{\lambda}}} (1 + \|u\|_{\tilde{L}^1([t_0, t); \dot{H}^2)}) e^{C \|\nabla u\|_{L^1([t_0, t); L^\infty)}} \leq C(t).$$

Hence we deduce from inequalities (3.22) and (3.23) that

$$\|(\partial_t u, \nabla \Pi)\|_{L_{\text{loc}}^1([t_0, +\infty[; \dot{B}_{p,1}^{-1+\frac{2}{p}})} \leq C(t).$$

This completes the existence part of Theorem 1.1.

Step 2. Uniqueness of strong solutions.

We divide the uniqueness part further into the following three sub-steps:

Step 2.1 Propagation of regularity of the density a for $p \in [2, \infty[$.

It follows from Theorem 3.14 of [7] that

$$\|a\|_{L_t^\infty(\dot{B}_{2,1}^1)} \leq \|a_0\|_{\dot{B}_{2,1}^1} e^{C\|u\|_{L_t^1(\dot{B}_{\infty,1}^1)}}.$$

When $p \in]2, \infty[$, we deduce from the transport equation of (1.1) that

$$\partial_t \partial_j \dot{\Delta}_q a + u \cdot \nabla \partial_j \dot{\Delta}_q a = -\dot{\Delta}_q (\partial_j u \cdot \nabla a) - [\dot{\Delta}_q, u \cdot \nabla] \partial_j a,$$

from which and $\operatorname{div} u = 0$, we infer

$$\|\dot{\Delta}_q \partial_j a(t)\|_{L^{\frac{p}{p-1}}} \leq \|\dot{\Delta}_q \partial_j a_0\|_{L^{\frac{p}{p-1}}} + \int_0^t \|\dot{\Delta}_q (\partial_j u \cdot \nabla a)\|_{L^{\frac{p}{p-1}}} d\tau + \int_0^t \|[\dot{\Delta}_q, u \cdot \nabla] \partial_j a\|_{L^{\frac{p}{p-1}}} d\tau.$$

Since $0 \leq 1 - \frac{2}{p} < 1$, we get, by applying classical commutator's estimate (see Lemma 2.100 and Remark 2.102 in [7]), that

$$(4.5) \quad \|\partial_j a\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \leq \|\partial_j a_0\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} + \|\partial_j u \cdot \nabla a\|_{\tilde{L}_t^1(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} + C \int_0^t \|\nabla u\|_{L^\infty} \|\partial_j a\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} d\tau.$$

It follows from Proposition 2.1, $\operatorname{div} u = 0$ and $p \in]2, \infty[$ that

$$\|T_{\partial_j u} \nabla a\|_{L_t^1(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \lesssim \int_0^t \|\nabla u\|_{L^\infty} \|\nabla a\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} d\tau$$

and

$$(4.6) \quad \begin{aligned} & \|T_{\nabla a} \partial_j u\|_{\tilde{L}_t^1(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} + \|R(\partial_j u, a)\|_{\tilde{L}_t^1(\dot{B}_{\frac{p}{p-1}, \infty}^{2-\frac{2}{p}})} \\ & \lesssim \|\nabla a\|_{L_t^\infty(\dot{B}_{\frac{2p}{p-2}, \infty}^{-\frac{2}{p}})} \|\nabla u\|_{\tilde{L}_t^1(\dot{H}^1)} \lesssim \|\nabla a\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \|\nabla u\|_{\tilde{L}_t^1(\dot{H}^1)}. \end{aligned}$$

By using Bony's decomposition and substituting the above estimates into (4.5), we deduce from (3.18) that for $t \leq T_1$,

$$\|\nabla a\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \lesssim \|\nabla a_0\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} + C \int_0^t \|\nabla u\|_{L^\infty} \|\nabla a\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} d\tau.$$

Applying Gronwall's inequality gives

$$(4.7) \quad \|\nabla a\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \lesssim \|\nabla a_0\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} e^{C\|\nabla u\|_{L_t^1(L^\infty)}} \quad \text{for } t \leq T_1.$$

On the other hand, we deduce from (4.6) that

$$\|T_{\nabla a} \partial_j u\|_{L_t^1(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} + \|R(\partial_j u, a)\|_{\tilde{L}_t^1(\dot{B}_{\frac{p}{p-1}, \infty}^{2-\frac{2}{p}})} \lesssim \int_0^t \|u\|_{\dot{H}^2} \|\nabla a\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} d\tau.$$

By inserting the above estimate into (4.5), we infer for $t > t_0$ with $t_0 \leq T_1$

$$(4.8) \quad \|\nabla a\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \lesssim \|\nabla a(t_0)\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} + \int_{t_0}^t (\|\nabla u\|_{L^\infty} + \|u\|_{\dot{H}^2}) \|\nabla a\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} d\tau.$$

We thus get, by applying Gronwall's inequality to (4.8), that for $t \geq t_0$

$$(4.9) \quad \|\nabla a\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \lesssim \|\nabla a(t_0)\|_{\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}}} e^{C(\|\nabla u\|_{L_t^1(L^\infty)} + \|u\|_{L_t^1(\dot{H}^2)})} \leq C(t),$$

where we used (4.4) in the last step.

Step 2.2 The uniqueness of solution in case $p = 2$.

We shall use the Lagrangian approach to prove the uniqueness (see [16, 25] for instance). Let $(\rho, u, \nabla \Pi)$ be a global solution of (1.1) obtained in Theorem 1.1. Then due to $u \in L^1_{loc}(\mathbb{R}^+; Lip)$, we can define the trajectory $X(t, y)$ of $u(t, x)$ by

$$\partial_t X(t, y) = u(t, X(t, y)), \quad X(0, y) = y,$$

which leads to the following relation between the Eulerian coordinates x and the Lagrangian coordinates y :

$$(4.10) \quad x = X(t, y) = y + \int_0^t u(\tau, X(\tau, y)) d\tau.$$

Moreover, we can take T to be so small that

$$(4.11) \quad \int_0^T \|\nabla u(t, \cdot)\|_{L^\infty} dt \leq \frac{1}{2}.$$

Then for $t \leq T$, $X(t, y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, is invertible with respect to y variables, and we denote $Y(t, \cdot)$ to be its inverse mapping. Let

$$\bar{u}(t, y) \stackrel{\text{def}}{=} u(t, X(t, y)) \quad \text{and} \quad \bar{\Pi}(t, y) \stackrel{\text{def}}{=} \Pi(t, X(t, y)).$$

Then similar to [16], one has

$$(4.12) \quad \bar{u} \in \tilde{L}_{loc}^\infty(\mathbb{R}^+; \dot{B}_{2,1}^0) \quad \text{and} \quad \partial^2 \bar{u}, \partial_t \bar{u}, \nabla \bar{\Pi} \in L^1_{loc}(\mathbb{R}^+; \dot{B}_{2,1}^0),$$

and

$$(4.13) \quad \partial_t \bar{u}(t, y) = (\partial_t u + u \cdot \nabla u)(t, x), \quad \partial_{x_i} u_j(t, x) = \sum_{k=1}^2 \partial_{y_k} \bar{u}_j(t, y) \partial_{x_i} y_k,$$

for $x = X(t, y)$, $y = Y(t, x)$.

Let $A(t, y) \stackrel{\text{def}}{=} (\nabla X(t, y))^{-1} = \nabla_x Y(t, x)$, then we have

$$(4.14) \quad \nabla_x u(t, x) = A(t, y)^T \nabla_y \bar{u}(t, y) \quad \text{and} \quad \operatorname{div}_x u(t, x) = \operatorname{div}(A(t, y) \bar{u}(t, y)),$$

and $(\bar{u}, \nabla_y \bar{\Pi})$ solves

$$(4.15) \quad \begin{cases} \rho_0 \partial_t \bar{u} - \Delta_y \bar{u} + \nabla_y \bar{\Pi} = \operatorname{div}((AA^T - Id) \nabla_y \bar{u}) + (Id - A)^T \nabla_y \bar{\Pi}, \\ \operatorname{div}_y \bar{u} = \operatorname{div}((Id - A) \bar{u}). \end{cases}$$

Before proceeding, we recall the following two lemmas from [16]:

Lemma 4.1. Let $p \in [1, \infty[$, then under the assumption that $\int_0^T \|\nabla_y \bar{u}\|_{\dot{B}_{p,1}^{\frac{2}{p}}} d\tau \leq c$ ($c = c(p) > 0$ is a constant), for all $t \in [0, T]$, one has

$$\|Id - A\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \lesssim \|\nabla_y \bar{u}\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}, \quad \|\partial_t A\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \lesssim \|\nabla_y \bar{u}\|_{\dot{B}_{p,1}^{\frac{2}{p}}}, \quad \|AA^T - Id\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \lesssim \|\nabla_y \bar{u}\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}.$$

Lemma 4.2. Let $p \in [1, \infty[$, and \bar{u}^1 and \bar{u}^2 be two vector fields satisfying $\int_0^T \|\nabla_y \bar{u}^i(\tau, \cdot)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} d\tau \leq c$ ($c = c(p) > 0$ is a constant and $i = 1, 2$) and $\delta \bar{u} \stackrel{\text{def}}{=} \bar{u}^2 - \bar{u}^1$, then for all $t \in [0, T]$, we have

$$\|A^2 - A^1\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \lesssim \|\nabla_y \delta \bar{u}\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} \quad \text{and} \quad \|\partial_t(A^2 - A^1)\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} \lesssim \|\nabla_y \delta \bar{u}\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}.$$

Now let $(\rho^i, u^i, \nabla \Pi^i)$, $i = 1, 2$, be two solutions of (1.1) which satisfy the regularity properties of Theorem 1.1. Let $(\bar{u}^i, A^i, \bar{\Pi}^i)$, $i = 1, 2$, be given by (4.10)-(4.13), we set

$$(\delta A, \delta \bar{u}, \nabla \delta \bar{\Pi}) \stackrel{\text{def}}{=} (A^2 - A^1, \bar{u}^2 - \bar{u}^1, \nabla \bar{\Pi}^2 - \nabla \bar{\Pi}^1).$$

Then it follows from (4.15) that the system for $(\delta\bar{u}, \nabla\delta\bar{\Pi})$ reads

$$(4.16) \quad \begin{cases} \partial_t\delta\bar{u} - (1+a_0)(\Delta_y\delta\bar{u} + \nabla_y\delta\bar{\Pi}) = \delta\bar{F}, \\ \operatorname{div}_y\delta\bar{u} = \nabla\delta\bar{u} : (Id - A^2) - \nabla u^1 : \delta A = \operatorname{div}_y((Id - A^2)\delta\bar{u} - \delta A\bar{u}^1) \stackrel{\text{def}}{=} \operatorname{div}_y g, \\ \Delta\Phi = \operatorname{div}_y g, \\ \delta\bar{u}|_{t=0} = 0, \end{cases}$$

where

$$\begin{aligned} \delta\bar{F} &\stackrel{\text{def}}{=} (1+a_0)(Id - A^2)^T \nabla_y \delta\bar{\Pi} - (1+a_0)(\delta A^T \nabla_y \bar{\Pi}^1) \\ &\quad + (1+a_0)\operatorname{div}_y((A^2(A^2)^T - Id)\nabla_y\delta\bar{u} + (A^2(A^2)^T - A^1(A^1)^T)\nabla_y\bar{u}_1). \end{aligned}$$

We get, by applying Proposition 2.3 with $p = 2$, that

$$(4.17) \quad \begin{aligned} \|\delta\bar{u}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^0)} + \|(\partial_t\delta\bar{u}, \nabla_y^2\delta\bar{u}, \nabla_y\delta\bar{\Pi})\|_{L_t^1(\dot{B}_{2,1}^0)} &\lesssim \|\delta\bar{F}\|_{L_t^1(\dot{B}_{2,1}^0)} + \|\nabla\operatorname{div}_y g\|_{L_t^1(\dot{B}_{2,1}^0)} \\ &\quad + \|\partial_t g\|_{L_t^1(\dot{B}_{2,1}^0)} + (2^m\sqrt{t} + 2^{2m}t)(\|\partial_t g\|_{L_t^1(L^2)} + \|\delta\bar{F}\|_{L_t^1(L^2)}) + \|\nabla\Phi\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^0) \cap L_t^1(\dot{B}_{2,1}^2)}. \end{aligned}$$

It follows from Lemmas 4.1 and 4.2, $\frac{1}{2} < \frac{1}{2} + \frac{1}{\lambda} \leq 1$, and the law of product in Besov spaces in Proposition 2.1 that

$$\begin{aligned} \|\delta\bar{F}\|_{L_t^1(\dot{B}_{2,1}^0)} &\lesssim (1 + \|a_0\|_{\dot{B}_{\lambda,\infty}^{\frac{2}{\lambda}} \cap L^\infty}) \left(\| (Id - A^2)^T \|_{L_t^\infty(\dot{B}_{2,1}^1)} \|\nabla_y\delta\bar{\Pi}\|_{L_t^1(\dot{B}_{2,1}^0)} \right. \\ &\quad + \|\delta A^T\|_{L_t^\infty(\dot{B}_{2,1}^1)} \|\nabla_y\bar{\Pi}^1\|_{L_t^1(\dot{B}_{2,1}^0)} + \|(A^2(A^2)^T - Id)\|_{L_t^\infty(\dot{B}_{2,1}^1)} \|\nabla_y\delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^1)} \\ &\quad \left. + \|(A^2(A^2)^T - A^1(A^1)^T)\|_{L_t^\infty(\dot{B}_{2,1}^1)} \|\nabla_y\bar{u}_1\|_{L_t^1(\dot{B}_{2,1}^1)} \right) \\ &\lesssim \|\nabla_y\bar{u}^2\|_{L_t^1(\dot{B}_{2,1}^1)} \|\nabla_y\delta\bar{\Pi}\|_{L_t^1(\dot{B}_{2,1}^0)} + \|(\nabla_y\bar{u}^1, \nabla_y\bar{u}^2)\|_{L_t^1(\dot{B}_{2,1}^1)} \|\nabla_y\delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^1)} \\ &\quad + \|\nabla_y\bar{\Pi}^1\|_{L_t^1(\dot{B}_{2,1}^0)} \|\nabla_y\delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^1)}, \end{aligned}$$

and

$$\begin{aligned} \|\nabla\operatorname{div}_y g\|_{L_t^1(\dot{B}_{2,1}^0)} &= \|\operatorname{div}_y g\|_{L_t^1(\dot{B}_{2,1}^1)} \\ &\lesssim \|\nabla_y\delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^1)} \|Id - A^2\|_{L_t^\infty(\dot{B}_{2,1}^1)} + \|\nabla_y u^1\|_{L_t^1(\dot{B}_{2,1}^1)} \|\delta A\|_{L_t^\infty(\dot{B}_{2,1}^1)} \\ &\lesssim \|(\nabla_y\bar{u}^1, \nabla_y\bar{u}^2)\|_{L_t^1(\dot{B}_{2,1}^1)} \|\nabla_y\delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^1)} \end{aligned}$$

and

$$\begin{aligned} \|\partial_t g\|_{L_t^1(\dot{B}_{2,1}^0)} &\lesssim \|\partial_t A^2 \delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^0)} + \|(Id - A^2) \partial_t \delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^0)} + \|\partial_t \delta A \bar{u}^1\|_{L_t^1(\dot{B}_{2,1}^0)} + \|\delta A \partial_t \bar{u}^1\|_{L_t^1(\dot{B}_{2,1}^0)} \\ &\lesssim \left(\|(\nabla^2\bar{u}^2, \partial_t\bar{u}^1)\|_{L_t^1(\dot{B}_{2,1}^0)} + \|\bar{u}^1\|_{L_t^\infty(\dot{B}_{2,1}^0)}^{\frac{1}{2}} \|\bar{u}^1\|_{L_t^1(\dot{B}_{2,1}^2)}^{\frac{1}{2}} \right) \\ &\quad \times \left(\|\delta\bar{u}\|_{L_t^\infty(\dot{B}_{2,1}^0)} + \|(\nabla^2\delta\bar{u}, \partial_t\delta\bar{u})\|_{L_t^1(\dot{B}_{2,1}^0)} \right). \end{aligned}$$

Observing that $\dot{B}_{2,1}^0 \hookrightarrow \dot{B}_{2,2}^0$, we have

$$\|\partial_t g\|_{L_t^1(L^2)} + \|\delta\bar{F}\|_{L_t^1(L^2)} \leq \|\partial_t g\|_{L_t^1(\dot{B}_{2,1}^0)} + \|\delta\bar{F}\|_{L_t^1(\dot{B}_{2,1}^0)}.$$

As $\Phi(0) = 0$, there holds $\Phi(t) = \int_0^t \partial_\tau \Phi(\tau) d\tau$, so that one has

$$\|\nabla\Phi\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^0)} \lesssim \|\partial_t g\|_{L_t^1(\dot{B}_{2,1}^0)}.$$

Finally let us turn to the estimate of $\|\nabla\Phi\|_{L_t^1(\dot{B}_{2,1}^2)}$. Indeed it follows from the law of product that

$$\begin{aligned} \|\nabla\Phi\|_{L_t^1(\dot{B}_{2,1}^2)} &\lesssim \|\operatorname{div}_y g\|_{L_t^1(\dot{B}_{2,1}^1)} \lesssim \|\nabla\delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^1)} \|Id - A^2\|_{L_t^\infty(\dot{B}_{2,1}^1)} + \|\nabla\bar{u}^1\|_{L_t^1(\dot{B}_{2,1}^1)} \|\delta A\|_{L_t^\infty(\dot{B}_{2,1}^1)} \\ &\lesssim \|\nabla\bar{u}^2\|_{L_t^1(\dot{B}_{2,1}^1)} \|\nabla\delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^1)} + \|\nabla\bar{u}^1\|_{L_t^1(\dot{B}_{2,1}^1)} \|\nabla\delta\bar{u}\|_{L_t^1(\dot{B}_{2,1}^1)}. \end{aligned}$$

By substituting the above estimates into (4.17), we obtain

$$\begin{aligned} & \|\delta\bar{u}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^0)} + \|(\partial_t\delta\bar{u}, \nabla_y^2\delta\bar{u}, \nabla_y\delta\bar{\Pi})\|_{L_t^1(\dot{B}_{2,1}^0)} \\ & \leq C(\|(\nabla_y^2\bar{u}^1, \nabla_y^2\bar{u}^2, \nabla_y\bar{\Pi}^1, \partial_t\bar{u}^1)\|_{L_t^1(\dot{B}_{2,1}^0)} + \|\bar{u}^1\|_{L_t^\infty(\dot{B}_{2,1}^0)}^{\frac{1}{2}} \|\bar{u}^1\|_{L_t^1(\dot{B}_{2,1}^1)}^{\frac{1}{2}}) \\ & \quad \times (\|\delta\bar{u}\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^0)} + \|(\partial_t\delta\bar{u}, \nabla_y^2\delta\bar{u}, \nabla_y\delta\bar{\Pi})\|_{L_t^1(\dot{B}_{2,1}^0)}). \end{aligned}$$

Then by taking t to be so small that

$$C(\|(\nabla_y^2\bar{u}^1, \nabla_y^2\bar{u}^2, \nabla_y\bar{\Pi}^1, \partial_t\bar{u}^1)\|_{L_t^1(\dot{B}_{2,1}^0)} + \|\bar{u}^1\|_{L_t^\infty(\dot{B}_{2,1}^0)}^{\frac{1}{2}} \|\bar{u}^1\|_{L_t^1(\dot{B}_{2,1}^1)}^{\frac{1}{2}}) < 1,$$

we deduce the uniqueness part of Theorem 1.1 for the case when $p = 2$.

Step 2.2 The uniqueness of solution in case $p \in]2, \infty[$.

Let $(\rho^i, u^i, \nabla\Pi^i)$, $i = 1, 2$, be two solutions of (1.1) which satisfy the regularity properties of Theorem 1.1. We set $\rho \stackrel{\text{def}}{=} \frac{1}{1+a}$ and denote

$$(\delta a, \delta u, \nabla\delta\Pi) \stackrel{\text{def}}{=} (a^2 - a^1, u^2 - u^1, \nabla\Pi^2 - \nabla\Pi^1).$$

Then in view of (1.1), the system for $(\delta a, \delta u, \nabla\delta\Pi)$ reads

$$(4.18) \quad \begin{cases} \partial_t\delta a + u^2 \cdot \nabla\delta a = -\delta u \cdot \nabla a^1, \\ \partial_t\delta u + (u^2 \cdot \nabla)\delta u - (1+a^2)(\Delta\delta u - \nabla\delta\Pi) = \delta F, \\ \operatorname{div} \delta u = 0, \\ (\delta a, \delta u)|_{t=0} = (0, 0), \end{cases}$$

where δF is determined by $\delta F \stackrel{\text{def}}{=} -(\delta u \cdot \nabla)u^1 + \delta a(\Delta u^1 - \nabla\Pi^1)$.

To estimate δu , for integer $k \in \mathbb{N}$, we first write the momentum equation of (4.18) as

$$(4.19) \quad \begin{aligned} & \partial_t\delta u + (u^2 \cdot \nabla)\delta u - (1+S_k a^2)(\Delta\delta u - \nabla\delta\Pi) = \delta\mathcal{F}_k \quad \text{with} \\ & \delta\mathcal{F}_k \stackrel{\text{def}}{=} (a^2 - S_k a^2)(\Delta\delta u - \nabla\delta\Pi) - \delta u \cdot \nabla u^1 + \delta a(\Delta u^1 - \nabla\Pi^1). \end{aligned}$$

Then we get, by applying Proposition 2.4 to (4.19), that for all $t \in]0, T]$,

$$(4.20) \quad \begin{aligned} \|\delta u\|_{L_t^\infty(B_{2,\infty}^{-1})} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} & \leq C e^{C(t+t\|\nabla S_k a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)}^2 + \|u^2\|_{L_t^1(B_{\infty,1}^1)})} \\ & \quad \times (\|\delta\mathcal{F}_k\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} + \|S_k \nabla a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)} \|\nabla\delta\Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})}). \end{aligned}$$

Notice that

$$\begin{aligned} \|\nabla S_k a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)} & \lesssim \|\nabla S_k a^2\|_{\tilde{L}_t^\infty(L^2)} + \|\nabla S_k a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)} \\ & \lesssim \|a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)} + 2^k \|a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)} \lesssim 2^k \|a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)}. \end{aligned}$$

By substituting the above estimate into (4.20) and using (4.3), we obtain

$$(4.21) \quad \|\delta u\|_{L_t^\infty(B_{2,\infty}^{-1})} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} \leq C e^{C t 2^k} (\|\nabla\delta\Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} + \|\delta\mathcal{F}_k\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})}).$$

On the other hand, we get, by applying div to the momentum equation of (4.18), that

$$(4.22) \quad \operatorname{div}((1+a^2)\nabla\delta\Pi) = \operatorname{div} G$$

with

$$G \stackrel{\text{def}}{=} (a^2 - S_m a^2)\Delta\delta u + S_m a^2 \Delta\delta u - \delta u \cdot \nabla u^1 - u^2 \cdot \nabla\delta u + \delta a(\Delta u^1 - \nabla\Pi^1) \stackrel{\text{def}}{=} \sum_{\ell=1}^5 I_\ell.$$

We deduce from Propositions 2.2 and 3.1 that, for any small constant $c_0 > 0$, there exist sufficiently large $j_0 \in \mathbb{N}$ and a positive existence time T_2 such that

$$(4.23) \quad \|a^2 - S_j a^2\|_{\tilde{L}_{T_2}^\infty(B_{2,1}^1)} \leq \|a^2 - \dot{S}_j a^2\|_{\tilde{L}_{T_2}^\infty(\dot{B}_{2,1}^1)} \leq c_0, \quad \text{for any } j \geq j_0.$$

Then we get, by applying Proposition 2.5 to (4.22), that

$$(4.24) \quad \|\nabla \delta\Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \lesssim (1 + 2^j \|a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)} (1 + \|a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)})) (\|G\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|\operatorname{div} G\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})}).$$

While it follows from Lemma 2.1 and product laws in Besov spaces in Proposition 2.1 that

$$\|I_1\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|\operatorname{div} I_1\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} \lesssim \|I_1\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \lesssim \|a^2 - S_m a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)} \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)}$$

and

$$\begin{aligned} & \|I_2\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|\operatorname{div} I_2\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} \\ & \lesssim \|T_{S_m a^2} \Delta \delta u\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|T_{\Delta \delta u} S_m a^2\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} \\ & \quad + \|R(S_m a^2, \Delta \delta u)\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} + \|T_{\nabla S_m a^2} \Delta \delta u\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|T_{\Delta \delta u} \nabla S_m a^2\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} \\ & \lesssim (\|S_m a^2\|_{L_t^\infty(L^\infty)} + 2^m \|\nabla S_m a^2\|_{L_t^\infty(B_{2,1}^0)}) \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^0)} \lesssim 2^m \|a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)} \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^0)}. \end{aligned}$$

Along the same line, one has

$$\begin{aligned} & \|(I_3, \operatorname{div} I_3)\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|I_4\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|\operatorname{div} I_4\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} \\ & \lesssim \|I_3\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} + \|T_{u^2} \nabla \delta u\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|T_{\nabla \delta u} u^2\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|R(u^2, \delta u)\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \\ & \quad + \|T_{\nabla u^2} \nabla \delta u\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|T_{\nabla \delta u} \nabla u^2\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|R(\partial_\ell u^2, \delta u_\ell)\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \\ & \lesssim \int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} (\|u^1\|_{B_{\infty,1}^1} + \|u^2\|_{B_{\infty,1}^1}) d\tau. \end{aligned}$$

To deal with I_5 , we write by using Bony's decomposition that

$$(4.25) \quad I_5 = \delta a(\Delta u^1 - \nabla \Pi^1) = T_{\Delta u^1 - \nabla \Pi^1} \delta a + T_{\delta a}(\Delta u^1 - \nabla \Pi^1) + R(\delta a, \Delta u^1 - \nabla \Pi^1).$$

It follows from product laws in Besov spaces in Proposition 2.1 again that

$$\begin{aligned} \|T_{\Delta u^1 - \nabla \Pi^1} \delta a\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} & \lesssim \int_0^t \|\delta a\|_{B_{2,\infty}^0} (\|\Delta u^1\|_{B_{\infty,\infty}^{-1}} + \|\nabla \Pi^1\|_{B_{\infty,\infty}^{-1}}) d\tau \\ & \lesssim \int_0^t \|\delta a\|_{B_{2,\infty}^0} (\|\Delta u^1\|_{B_{p,1}^{-1+\frac{2}{p}}} + \|\nabla \Pi^1\|_{B_{p,1}^{-1+\frac{2}{p}}}) d\tau, \end{aligned}$$

and

$$\|T_{\delta a}(\Delta u^1 - \nabla \Pi^1)\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \lesssim \int_0^t \|\delta a\|_{B_{2,\infty}^0} (\|\Delta u^1\|_{B_{p,1}^{-1+\frac{2}{p}}} + \|\nabla \Pi^1\|_{B_{p,1}^{-1+\frac{2}{p}}}) d\tau,$$

and

$$\begin{aligned} \|R(\delta a, \Delta u^1 - \nabla \Pi^1)\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} & \lesssim \|R(\delta a, \Delta u^1 - \nabla \Pi^1)\|_{L_t^1(B_{1,\infty}^0)} \\ & \lesssim \int_0^t \|\delta a\|_{B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}}} (\|\Delta u^1\|_{B_{p,1}^{-1+\frac{2}{p}}} + \|\nabla \Pi^1\|_{B_{p,1}^{-1+\frac{2}{p}}}) d\tau. \end{aligned}$$

Since $p > 2$, $\dot{B}_{p,1}^{-1+\frac{2}{p}} \hookrightarrow B_{p,1}^{-1+\frac{2}{p}}$, as a consequence, we have

$$\|(I_5, \operatorname{div} I_5)\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} \lesssim \|\delta a(\Delta u^1 - \nabla \Pi^1)\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \lesssim \int_0^t \|\delta a\|_{B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}}} \|(\Delta u^1, \nabla \Pi^1)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} d\tau.$$

where we used the fact that ($p \geq 2$)

$$\|\Delta u^1\|_{B_{p,1}^{-1+\frac{2}{p}}} + \|\nabla \Pi^1\|_{B_{p,1}^{-1+\frac{2}{p}}} \lesssim \|\Delta u^1\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \|\nabla \Pi^1\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}.$$

By summarizing the above estimates, we obtain

$$\begin{aligned} \|G\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} + \|\operatorname{div} G\|_{\tilde{L}_t^1(B_{2,\infty}^{-2})} &\lesssim \|a^2 - S_m a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)} \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} + 2^m \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^0)} \\ &+ \int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} \|(u^1, u^2)\|_{B_{\infty,1}^1} d\tau + \int_0^t \|\delta a\|_{B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}}} \|(\Delta u^1, \nabla \Pi^1)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} d\tau. \end{aligned}$$

By substituting the above estimates into (4.24), we arrive at

$$\begin{aligned} \|\nabla \delta \Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} &\lesssim (1 + 2^j \|a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)} (1 + \|a^2\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^1)})) \\ (4.26) \quad &\times \left(\|a^2 - S_m a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)} \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} + 2^m \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^0)} \right. \\ &\left. + \int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} \|(u^1, u^2)\|_{B_{\infty,1}^1} d\tau + \int_0^t \|\delta a\|_{B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}}} \|(\Delta u^1, \nabla \Pi^1)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} d\tau \right). \end{aligned}$$

To handle the estimate of $\|\delta \mathcal{F}_k\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})}$, we get, by applying the law of product in Besov spaces, Proposition 2.1, that

$$\begin{aligned} \|\delta \mathcal{F}_k\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} &\lesssim \|a^2 - S_k a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)} (\|\Delta \delta u\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})}) \\ (4.27) \quad &+ \int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} \|u^1\|_{B_{\infty,1}^1} d\tau + \int_0^t \|\delta a\|_{B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}}} (\|u^1\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}} + \|\nabla \Pi^1\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}) d\tau. \end{aligned}$$

For $m, k \geq j_0$ and $t \leq T_2$, we get, by substituting (4.23), (4.26) and (4.27) into (4.21), that

$$\begin{aligned} &\|\delta u\|_{L_t^\infty(B_{2,\infty}^{-1})} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \\ (4.28) \quad &\leq C e^{C t 2^k} (2^j \|a^2 - S_m a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)} + \|a^2 - S_k a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)}) (\|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})}) \\ &+ C e^{C t 2^k} 2^j \left(\int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} \|(u^1, u^2)\|_{B_{\infty,1}^1} d\tau + \int_0^t \|\delta a\|_{B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}}} \|(\Delta u^1, \nabla \Pi^1)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} d\tau \right) \\ &+ C e^{C t 2^k} 2^{m+j} \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^0)}. \end{aligned}$$

Notice that $\|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^0)} \lesssim \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})}^{\frac{1}{2}} \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)}^{\frac{1}{2}}$, we infer

$$(4.29) \quad C e^{C t 2^k} 2^{m+j} \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^0)} \leq \frac{1}{2} \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} + C e^{C t 2^k} 2^{2m+2j} \int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} d\tau,$$

On the other hand, in view of the first equation in (4.18), we deduce from the classical estimate of the transport equation, that

$$\|\delta a\|_{L_t^\infty(B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}})} \leq \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}})} e^{C \|\nabla u^2\|_{L_t^1(L^\infty)}} \lesssim \|\delta u \cdot \nabla a^1\|_{L_t^1(B_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}})}.$$

Thanks to Proposition 2.1, one has

$$\begin{aligned} \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(\dot{B}_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}})} &\lesssim (\|\delta u\|_{L_t^1(L^\infty)} + \|\delta u\|_{\tilde{L}_t^1(\dot{B}_{2,1}^1)}) \|\nabla a^1\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}})}, \\ \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(\dot{B}_{\frac{p}{p-1},1}^0)} &\lesssim \|\delta u\|_{L_t^1(\dot{B}_{2,1}^{\frac{2}{p}})} \|\nabla a^1\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1},\infty}^{1-\frac{2}{p}})}, \end{aligned}$$

which along with (2.3) yields

$$\begin{aligned} \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(B_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} &\lesssim \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(\dot{B}_{\frac{p}{p-1}}^{1-\frac{2}{p}})} + \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(\dot{B}_{\frac{p}{p-1}, 1}^0)} \\ &\lesssim (\|\delta u\|_{L_t^1(L^\infty)} + \|\delta u\|_{\tilde{L}_t^1(\dot{B}_{2,\infty}^1)} + \|\delta u\|_{L_t^1(\dot{B}_{2,1}^{\frac{2}{p}})}) \|\nabla a^1\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \\ &\lesssim (\|\delta u\|_{L_t^1(L^\infty)} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)}) \|\nabla a^1\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} (4.30) \quad \|\delta a\|_{L_t^\infty(B_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} &\lesssim (\|\delta u\|_{L_t^1(L^\infty)} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)}) \|\nabla a^1\|_{L_t^\infty(\dot{B}_{\frac{p}{p-1}, \infty}^{1-\frac{2}{p}})} \\ &\lesssim \|\delta u\|_{L_t^1(L^\infty)} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)}. \end{aligned}$$

Inserting (4.29) and (4.30) into (4.28) yields

$$\begin{aligned} (4.31) \quad &\|\delta u\|_{L_t^\infty(B_{2,\infty}^{-1})} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \\ &\leq C e^{C t 2^k} (2^j \|a^2 - S_m a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)} + \|a^2 - S_k a^2\|_{\tilde{L}_t^\infty(B_{2,1}^1)}) (\|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})}) \\ &\quad + C e^{C t 2^k} 2^j \left(\int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} \|(u^1, u^2)\|_{B_{\infty,1}^1} d\tau \right. \\ &\quad \left. + \int_0^t (\|\delta u\|_{L_\tau^1(L^\infty)} + \|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)}) \|(\Delta u^1, \nabla \Pi^1)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} d\tau \right) \\ &\quad + C e^{C t 2^k} 2^{2m+2j} \int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} d\tau. \end{aligned}$$

Therefore, for given $m, k \geq j_0$, taking $c_0 > 0$ in (4.23) small enough, we deduce from (4.23) that any $t \leq T_2$,

$$\begin{aligned} (4.32) \quad &\|\delta u\|_{L_t^\infty(B_{2,\infty}^{-1})} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(B_{2,\infty}^{-1})} \\ &\leq C e^{C t 2^k} 2^j \left(\int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} \|(u^1, u^2)\|_{B_{\infty,1}^1} d\tau \right. \\ &\quad \left. + \int_0^t (\|\delta u\|_{L_\tau^1(L^\infty)} + \|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)}) \|(\Delta u^1, \nabla \Pi^1)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} d\tau \right) \\ &\quad + C e^{C t 2^k} 2^{2m+2j} \int_0^t \|\delta u\|_{B_{2,\infty}^{-1}} d\tau. \end{aligned}$$

Then for $T_3 \in (0, T_2]$ being small enough, we deduce that for all $t \in [0, T_3]$,

$$\begin{aligned} (4.33) \quad &\|\delta u\|_{L_t^\infty(B_{2,\infty}^{-1})} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} \\ &\lesssim \int_0^t (\|\delta u\|_{L_\tau^1(L^\infty)} + \|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)}) (\|u^1\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}} + \|\nabla \Pi^1\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}) d\tau. \end{aligned}$$

Let N be an arbitrary positive integer which will be determined later on, we write

$$\|\delta u\|_{L_\tau^1(L^\infty)} \leq \|\delta u\|_{L_\tau^1(\dot{B}_{\infty,1}^0)} \leq \left(\sum_{-1 \leq q \leq N} + \sum_{N+1 \leq q \leq 2N} + \sum_{q \geq 2N+1} \right) \|\Delta_q \delta u\|_{L_\tau^1(L^\infty)},$$

from which and Lemma 2.1, we infer

$$\|\delta u\|_{L_\tau^1(L^\infty)} \lesssim 2^N \|\delta u\|_{L_\tau^1(L^2)} + N \|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)} + 2^{-N} \|\nabla \delta u\|_{\tilde{L}_\tau^1(L^\infty)}.$$

If we choose N so that

$$N \sim \ln \left(e + \frac{\|\delta u\|_{L_\tau^1(L^2)} + \|\nabla \delta u\|_{L_\tau^1(L^\infty)}}{\|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)}} \right),$$

we obtain

$$\begin{aligned} \|\delta u\|_{L_\tau^1(L^\infty)} &\lesssim \|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)} \ln \left(e + \frac{\|\delta u\|_{L_\tau^1(L^2)} + \|\nabla \delta u\|_{L_\tau^1(L^\infty)}}{\|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)}} \right) \\ (4.34) \quad &\lesssim \|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)} \ln \left(e + \sum_{i=1}^2 \frac{\tau \|u^i\|_{L_t^\infty(L^2)} + \|\nabla u^i\|_{L_\tau^1(L^\infty)}}{\|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)}} \right). \end{aligned}$$

Notice that for $\alpha \geq 0$ and $x \in (0, 1]$, there holds

$$\ln(e + \alpha x^{-1}) \leq \ln(e + \alpha)(1 - \ln x) \quad \text{and} \quad x \leq x(1 - \ln x).$$

Then by plugging (4.34) into (4.33), we find

$$\begin{aligned} &\|\delta u\|_{L_t^\infty(B_{2,\infty}^{-1})} + \|\delta u\|_{\tilde{L}_t^1(B_{2,\infty}^1)} \\ (4.35) \quad &\lesssim \int_0^t \|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)} (1 - \ln \|\delta u\|_{\tilde{L}_\tau^1(B_{2,\infty}^1)}) (\|u^1\|_{\dot{B}_{p,1}^{1+\frac{2}{p}}} + \|\nabla \Pi^1\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}}) d\tau. \end{aligned}$$

As $\int_0^1 \frac{dx}{x(1-\ln x)} = +\infty$, and $\|\Delta u^1\|_{L^2} + \|\nabla \Pi^1\|_{L^2}$ is locally integral in $t \in \mathbb{R}^+$, we deduce from Osgood's Lemma (see [22] for instance) that $\delta u(t) = 0$ for $t \leq T_3$, which together with (4.30) and (4.26) implies that $\delta a(t) = \delta \nabla \Pi(t) = 0$ for all $t \in [0, T_3]$.

The uniqueness of such solutions on the whole time interval $[0, +\infty)$ then follows by a bootstrap argument, which completes the proof of Theorem 1.1. \square

Finally let us present the proof of Corollary 1.1.

Proof of Corollary 1.1. We first deduce from Theorem 1.1 that the system (1.1) has a unique global solution (ρ, u) so that

$$\begin{aligned} \rho^{-1} - 1 &\in C([0, \infty[; \dot{B}_{\frac{2p}{2-p}, 2}^{-1+\frac{2}{p}} \cap L^\infty) \cap L^\infty(\mathbb{R}_+; L^\infty), \\ u &\in C([0, \infty[; \dot{B}_{2,1}^0) \cap L_{loc}^1(\mathbb{R}_+; \dot{B}_{2,1}^2), \\ \nabla \Pi &\in L_{loc}^1(\mathbb{R}_+; \dot{B}_{2,1}^0) \quad \text{and} \quad \partial_t u \in L_{loc}^1(\mathbb{R}_+; \dot{B}_{2,1}^0). \end{aligned}$$

It follows from (3.9) that

$$\begin{aligned} \|\delta u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} &\lesssim \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} \|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L_t^1(L^p)} \\ (4.36) \quad &+ \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} \|[\dot{\Delta}_q \mathbb{P}, \rho^{-1}] (\Delta u - \nabla \Pi)\|_{L_t^1(L^p)}. \end{aligned}$$

By applying classical commutator's estimate (see Lemma 2.100 and Remark 2.102 in [7]), we have

$$(4.37) \quad \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} \|[\dot{\Delta}_q \mathbb{P}, u \cdot \nabla] u\|_{L_t^1(L^p)} \lesssim \int_0^t \|\nabla u(\tau)\|_{L^\infty} \|u(\tau)\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} d\tau.$$

To handle the last term in (4.36), we write, by using homogeneous Bony's decomposition, that (with $f = \Delta u - \nabla \Pi$)

$$(4.38) \quad [\dot{\Delta}_q \mathbb{P}, a] f = \dot{\Delta}_q \mathbb{P} T_f a + \dot{\Delta}_q \mathbb{P} R(a, f) - T'_{\dot{\Delta}_q \mathbb{P} f} a - [\dot{\Delta}_q \mathbb{P}, T_a] f.$$

Due to $p \in]1, 2[$, it follows from Lemma 2.1 and the law of product in Besov spaces that

$$\begin{aligned} \|\dot{\Delta}_q \mathbb{P} T_f a\|_{L_t^1(L^p)} &\lesssim d_q 2^{(1-\frac{2}{p})q} \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\frac{2p}{2-p}, 1}^{-1+\frac{2}{p}})} \|f\|_{L_t^1(L^2)}, \\ \|\dot{\Delta}_q \mathbb{P} R(a, f)\|_{L_t^1(L^p)} &\lesssim d_q 2^{(1-\frac{2}{p})q} \|a\|_{L_t^\infty(\dot{B}_{\frac{2p}{2-p}, \infty}^{-1+\frac{2}{p}})} \|f\|_{L_t^1(\dot{B}_{2,1}^0)}. \end{aligned}$$

While we observe that

$$\|T'_{\dot{\Delta}_q \mathbb{P} f} a\|_{L_t^1(L^p)} \lesssim \sum_{k \geq q-3} \|\dot{S}_{k+2} \dot{\Delta}_q f\|_{L_t^1(L^2)} \|\dot{\Delta}_k a\|_{L_t^\infty(L^{\frac{2p}{2-p}})} \lesssim d_q 2^{(1-\frac{2}{p})q} \|a\|_{L_t^\infty(\dot{B}_{\frac{2p}{2-p}, \infty}^{-1+\frac{2}{p}})} \|f\|_{L_t^1(\dot{B}_{2,1}^0)}.$$

Finally, it follows from Lemma 2.2 that

$$\begin{aligned} \|[\dot{\Delta}_q \mathbb{P}, T_a] f\|_{L_t^1(L^p)} &\lesssim \sum_{|q-k| \leq 4} 2^{-q} \|\nabla \dot{S}_{k-1} a\|_{L_t^\infty(L^{\frac{2p}{2-p}})} \|\dot{\Delta}_k f\|_{L_t^1(L^2)} \\ &\lesssim d_q 2^{q(1-\frac{2}{p})} \|a\|_{L_t^\infty(\dot{B}_{\frac{2p}{2-p}, \infty}^{-1+\frac{2}{p}})} \|f\|_{L_t^1(\dot{B}_{2,1}^0)}. \end{aligned}$$

By summarizing the above estimates, we arrive at

$$(4.39) \quad \sum_{q \in \mathbb{Z}} 2^{q(\frac{2}{p}-1)} \|[\dot{\Delta}_q \mathbb{P}, a] f\|_{L_t^1(L^p)} \lesssim \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\frac{2p}{2-p}, 1}^{-1+\frac{2}{p}})} \|f\|_{L_t^1(\dot{B}_{2,1}^0)}.$$

By substituting the estimates (4.37) and (4.39) and then applying Gronwall's inequality, we achieve

$$\|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{-1+\frac{2}{p}}) \cap L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} \lesssim (\|u_0\|_{\dot{B}_{p,1}^{-1+\frac{2}{p}}} + \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\frac{2p}{2-p}, 1}^{-1+\frac{2}{p}})} (\|u\|_{L_t^1(\dot{B}_{2,1}^2)} + \|\nabla \Pi\|_{L_t^1(\dot{B}_{2,1}^0)})) e^{C\|\nabla u\|_{L_t^1(L^\infty)}}.$$

While it follows from Theorem 3.14 of [7] that

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}_{\frac{2p}{2-p}, 1}^{-1+\frac{2}{p}})} \lesssim \|a_0\|_{\dot{B}_{\frac{2p}{2-p}, 1}^{-1+\frac{2}{p}}} e^{C\|u\|_{L_t^1(\dot{B}_{\infty,1}^1)}}.$$

As $p > 1$, we deduce from the inequality (3.21) that

$$\|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim \|u \otimes u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + \|a \Delta u\|_{L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})} + \sum_{q \in \mathbb{Z}} 2^{q(-1+\frac{2}{p})} \|[\dot{\Delta}_q, a] \nabla \Pi\|_{L_t^1(L^p)},$$

from which and (4.39), we infer

$$\|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim \|u\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})}^2 + \|a\|_{\tilde{L}_t^\infty(\dot{B}_{\frac{2p}{2-p}, 1}^{-1+\frac{2}{p}})} (\|u\|_{L_t^1(\dot{B}_{p,1}^{1+\frac{2}{p}})} + \|\nabla \Pi\|_{L_t^1(\dot{B}_{2,1}^0)}).$$

Similarly, we deduce from the momentum equation of (1.3) that $\partial_t u \in L_t^1(\dot{B}_{p,1}^{-1+\frac{2}{p}})$. This completes the proof of Corollary 1.1. \square

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