LOW REGULARITY GLOBAL WELL-POSEDNESS OF AXISYMMETRIC MHD EQUATIONS WITH VERTICAL DISSIPATION AND MAGNETIC DIFFUSION

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Abstract. Consideration in this paper is the global well-posedness for the 3D axisymmetric MHD equations with only vertical dissipation and vertical magnetic diffusion. The existence of unique low regularity global solutions of the system with initial data in Lorentz spaces is established by using higher-order energy estimates and real interpolation method.

Keywords: Axisymmetric MHD equations; Global well-posedness; Lorentz spaces

AMS Subject Classification: 35Q30, 76D05

1. Introduction

We consider herein the 3-D incompressible anisotropic MHD equations

$$
\begin{cases}
\partial_t u + u\cdot\nabla u - (\nu_h\Delta_h + \nu_z\partial_z^2)u + \nabla\Pi = B\cdot\nabla B \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3,\\
\partial_t B + u\cdot\nabla B - (\mu_h\Delta_h + \mu_z\partial_z^2)B = B\cdot\nabla u,\\
\operatorname{div} u = \operatorname{div} B = 0,\\
(u,b)|_{t=0} = (u_0, B_0),
\end{cases}
\tag{1.1}
$$

where the unknowns u, B and Π represent the velocity of the fluid, the magnetic field and the scalar pressure function, respectively. The nonnegative constants ν_z (or ν_h) and μ_z (or μ_h) are the vertical (or horizontal) kinematic viscosity coefficient and magnetic diffusive coefficient. In (1.1), the usual Laplacians in the classical MHD equations are substituted by the anisotropic Laplacians $\nu_h\Delta_h + \nu_z\partial_z^2$ and $\mu_h\Delta_h + \mu_z\partial_z^2$.

The classical 3-D incompressible MHD equations

$$
\begin{cases}
\partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla\Pi = B\cdot\nabla B \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3,\\
\partial_t B + u\cdot\nabla B - \mu\Delta B = B\cdot\nabla u,\\
\operatorname{div} u = \operatorname{div} B = 0,\\
(u,b)|_{t=0} = (u_0, B_0),
\end{cases}
\tag{1.2}
$$

described the motion of electrically conducting fluids (e.g., astrophysics, geophysics, plasma physics and cosmology, see [7, 12, 26, 16]). The existence, uniqueness and regularity of system (1.2) has been extensively studied by many mathematicians recently. For the case that $\nu > 0$ and $\mu > 0$, it is well-known that Duvaut and Lions [13] proved the local existence and uniqueness of solutions to the d-D MHD system in the Sobolev space $H^s(\mathbb{R}^d), s \geq d$. They also obtained the global existence of solutions under the condition for small initial data. Later on, the global well-posedness of the 2-D MHD system with large initial data has been established by Sermange and Teman [27]. However, in the case in which ν and μ are all zero (i.e., the ideal MHD equations), the global well-posedness for the ideal MHD system remains a challenging open problem. Consequently, on the one hand, focuses have been on the equilibrium state for the MHD system (1.1) with partial dissipation [4, 33]. On the other hand, there are some mathematical papers [17, 31, 34] devoting to the global existence of the MHD system some partial regularity results and Serrin-type regularity criteria.

Note that when the initial magnetic field B_0 is identically zero, the system (1.1) reduces to the following 3-D incompressible anisotropic Navier-Stokes system

$$
\begin{cases}
\partial_t u + u\cdot\nabla u - (\nu_h\Delta_h + \nu_z\partial_z^2)u + \nabla\Pi = 0 \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3,\\
\operatorname{div} u = 0,\\
u|_{t=0} = u_0,
\end{cases}
\tag{1.3}
$$

which has been extensively studied by many mathematicians recently (see [8],[18],[24],[9], [15] etc.). In particular, for the case where $\nu_h > 0$ and $\nu_z = 0$ the system (1.3) has been studied for the first time by J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier in [8]. More precisely, the authors have proved in [8] the local in time existence of the solution when the initial data belongs to the anisotropic Sobolev space $H^{0,\frac{1}{2}+}$. The global well-posedness was proved for initial data which are small enough compared with horizontal viscosity ν_h, and moreover, the uniqueness of the solution was proved for more regular initial data, belonging to the space $H^{0,\frac{3}{2}+}$, which was removed later by D. Iftimie [18]. The critical case $s = \frac{1}{2}$ was studied by M. Paicu [24], who proved that the system (1.3) is locally well posed in the anisotropic Besov space $\dot{B}^{0,\frac{1}{2}}$, and the global existence of the solution was proved for small initial data compared with ν_h. Furthermore, J.-Y. Chemin and P. Zhang [9] obtained a similar result by working in an anisotropic Besov space with negative regularity indexes in the horizontal variable, which allows them to prove the global existence of the solution for horizontal Navier-Stokes equations with highly oscillating initial data in the horizontal variables. We recall that the main idea in the case where $\nu_h > 0$ and $\nu_z = 0$, in order to control the vertical derivative was to use the incompressibility condition, namely $\partial_x u^1 + \partial_y u^2 + \partial_z u^3 = 0$, which allows one to obtain a regularizing effect for the vertical component u^3 by using the horizontal viscosity.

Contrarily to the above situation, the case $\nu_h > 0$ and $\nu_v = 0$ is more difficult to study because of the lack of regularity in two horizontal variables. In fact, utilizing a regularizing effect only in the vertical direction seems very difficult to recover any regularization in all variables in the general case. For this reason, many mathematicians turn to studying the well-posedness of some particular cases, axisymmetric flows for example.

The vector field $u = u(x_1, x_2, z)$ is axisymmetric ("without swirl", i.e. $u^\theta \equiv 0$), if and only if, u^r and u^z do not depend on θ and

$$u = u^r(r,z)e_r + u^z(r,z)e_z,$$

where

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \quad e_z = (0,0,1), \quad r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan\frac{x_2}{x_1}.$$

A scalar function is called axisymmetric if it has no dependencies on the angular variable θ.

Indeed, for axisymmetric solutions, we have $\operatorname{div} u = \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0$. In the case without swirl, Ukhovskii and Yudovich [29] studied the global regularity of weak solutions of the axisymmetric Navier-Stokes equations applying the global regularity of the vorticity and the global *a priori* estimate $\|r^{-1}\omega\|_{L^r} \le \|r^{-1}\omega_0\|_{L^r}$ for $r \in [1, +\infty]$. Later on, Leonardi et al. [20] and Abidi [1] independently weakened the regularity assumption for $u_0 \in H^2(\mathbb{R}^3)$ and $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$. Furthermore, Abidi and Paicu [3] improved the regularity assumption to $\omega_0 \in L^{\frac{3}{2},1}(\mathbb{R}^3)$ and $r^{-1}\omega_0 \in L^{\frac{3}{2},1}(\mathbb{R}^3)$. The recent breakthrough is from Elgindi [14] on the singularity formation of the 3D Euler equation without swirl with $C^{1,\alpha}$ initial data for the velocity. Other results of axisymmetric Navier-Stokes equations can be found in [10, 32, 36].

Similarly, the axisymmetric "without swirl" MHD system in this paper means that the solution of the system (1.2) has the form

$$u(t, x_1, x_2, z) = u^r(t,r,z)e_r + u^z(t,r,z)e_z, \quad B(t, x_1, x_2, z) = B^\theta(t,r,z)e_\theta. \tag{1.4}$$

The global well-posedness of the axisymmetric "without swirl" MHD equations (1.2) (with $\nu>0$ and $\mu=0$) was established by Lei [19] for the initial data $(u_0,B_0)\in H^s(\mathbb{R}^3)$, $s\geq 2$ and $\frac{B_0^\theta}{r}\in L^\infty$. Recently, Ai and Li [5] weakened the condition to $(u_0,B_0)\in H^1(\mathbb{R}^3)\times H^2(\mathbb{R}^3)$ and $\frac{\omega_0}{r}\in L^2$. For regularity criteria for the axisymmetric MHD solutions, one may refer to [21, 30, 35] and the references cited therein.

Consider the case that the anisotropic Laplacians $\nu_h\Delta_h+\nu_z\partial_z^2$ and $\mu_h\Delta_h+\mu_z\partial_z^2$ have only vertical viscosity and magnetic diffusion, that is, $\nu_h=\mu_h=0$ and $\nu_z>0$, $\mu_z>0$, the system (1.1) reads as

$$\begin{cases}\partial_t u+u\cdot\nabla u-\nu_z\partial_z^2 u+\nabla\Pi=B\cdot\nabla B \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3,\\ \partial_t B+u\cdot\nabla B-\mu_z\partial_z^2 B=B\cdot\nabla u,\\ \operatorname{div} u=\operatorname{div} B=0,\\ (u,B)|_{t=0}=(u_0,B_0),\end{cases}\tag{1.5}$$

and its corresponding axisymmetric "without swirl" MHD system can be rewritten as

$$\begin{cases}\partial_t u^r+(u^r\partial_r+u^z\partial_z)u^r+\partial_r\Pi-\partial_z^2 u^r=-\frac{(B^\theta)^2}{r},\\ \partial_t u^z+(u^r\partial_r+u^z\partial_z)u^z+\partial_z\Pi-\partial_z^2 u^z=0,\\ \partial_t B^\theta+(u^r\partial_r+u^z\partial_z)B^\theta-\partial_z^2 B^\theta=\frac{u^rB^\theta}{r},\\ \partial_r u^r+\frac{u^r}{r}+\partial_z u^z=0.\end{cases}\tag{1.6}$$

For the initial data $(u_0,B_0)\in H^2(\mathbb{R}^3)$, and $\frac{B_0^\theta}{r}\in L^2\cap L^\infty(\mathbb{R}^3)$, Wang and Guo [30] established the existence of the unique global axisymmetric solutions to the system (1.5).

We remark that in previous works, well-posedness results were established for the initial data with high regularity. With the high regular initial data, the Lipschitz norm of the velocity u is locally integrable with respect to time t in $\mathbb{R}^+$, which ensures the propagation of the regularity of the solution. A natural and important question is whether a corresponding well-posedness result can be obtained for low regularity data. This kind of result may be helpful to understand the possible blow-up mechanism of the solution to the system (1.5), and shows that the model is applicable for general data without high regularity.

Our aim is to establish a family of low regularity global unique solutions to the axisymmetric "without swirl" MHD equations (1.5). Notice that the vorticity $\nabla\times u=\omega^\theta e_\theta$ with $\omega^\theta\stackrel{\text{def}}{=}\partial_z u^r-\partial_r u^z$. Setting $\omega\stackrel{\text{def}}{=}\omega^\theta$ and $b\stackrel{\text{def}}{=}B^\theta$, we know from (1.6) that $(\omega,\,b)$ satisfies

$$\begin{cases}\partial_t\omega+(u\cdot\nabla)\omega-\partial_z^2\omega=-\partial_z(\frac{b^2}{r})+\frac{u^r}{r}\omega,\\ \partial_t b+(u\cdot\nabla)b-\partial_z^2 b=\frac{u^r}{r}\,b,\\ u=(-\Delta)^{-1}\nabla\times(\omega\, e_\theta),\end{cases}\tag{1.7}$$

where the operator $u\cdot\nabla\stackrel{\text{def}}{=}u^r\partial_r+u^z\partial_z$.

Our main result is given as follows.

Theorem 1.1. *Let the initial data (ω_0,b_0) satisfy*

$$\omega_0,\ r^{-1}\omega_0\in L^{\frac{3}{2},1},\quad b_0\in L^{3,2},\quad r^{-1}b_0\in L^{\frac{3}{2},1}\cap L^{3,2}.\tag{1.8}$$

Let u_0 be an axisymmetric solenoidal vector-field with vorticity $\omega_0 e_\theta$ which is given by the Biot-Savart law:

$$u_0(X)=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(X-Y)\times(\omega_0e_\theta)(Y)}{|X-Y|^3}\,dY,$$

and B_0 be an axisymmetric solenoidal vector-field with the form $B_0=b_0\,e_\theta$. Then, the system (1.5) has a global in time axisymmetric solution (u,B) such that the vorticity ω and the magnetic field

be_θ satisfy

$$\omega,\, r^{-1}\omega,\, r^{-1}b \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2},1}(\mathbb{R}^3)\big), \quad \partial_z\omega,\, r^{-1}\partial_z\omega,\, r^{-1}\partial_z b \in L^2_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2},1}(\mathbb{R}^3)\big),$$
$$b,\, r^{-1}b \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^{3,2}(\mathbb{R}^3)\big).$$

Moreover, if, in addition, the initial data (ω_0, b_0) satisfies

$$\omega_0 \in L^{3,1}, \quad \partial_r\omega_0 \in L^{\frac{3}{2}}, \quad b_0,\, r^{-1}b_0 \in \dot{H}^1, \tag{1.9}$$

then the vorticity ω and the magnetic field be_θ also satisfy

$$\omega \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^{3,1}(\mathbb{R}^3)\big), \quad \partial_r\omega \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2}}(\mathbb{R}^3)\big), \quad \partial_z\partial_r\omega \in L^2_{loc}\big(\mathbb{R}_+;\, L^{\frac{3}{2}}(\mathbb{R}^3)\big),$$
$$(\nabla b, \nabla\frac{b}{r}) \in L^\infty_{loc}\big(\mathbb{R}_+;\, L^2(\mathbb{R}^3)\big), \quad (\partial_z\nabla b, \partial_z\nabla\frac{b}{r}) \in L^2_{loc}\big(\mathbb{R}_+;\, L^2(\mathbb{R}^3)\big),$$

and the solution is unique.

Remark 1.1. *Theorem 1.1 coincides with the primary conclusion of the Navier-Stokes equations in* [3] *if the initial magnetic field $B_0 \equiv 0$. In comparison to the result in* [30]*, the uniqueness of solutions to the MHD equations* (1.6) *with the low-regularity initial data in Theorem 1.1 is more challenging due to the lack of the control about the velocity u in $L^1_{loc}(\mathbb{R}^+; Lip)$.*

The proof of Theorem 1.1 is completed in Section 4. We now present a summary of the principal difficulties we encounter in our analysis as well as a sketch of the key ideas used in our proof.

To obtain the existence and uniqueness of regular solutions of the system (1.5), we need to establish some higher-order estimates of the velocity field for all $T > 0$. Since the system is degenerate along the horizontal direction, it is necessary to establish some new *a priori* estimates that overcome the difficulties caused by the lack of smoothing effects in the horizontal direction. To achieve this, some high-order estimates should be obtained from the system (1.7) about b and the vorticity $\nabla\times u = \omega e_\theta$ with $\omega = \partial_z u^r - \partial_r u^z$. As in the study of the 3-D axisymmetric Euler equations, for the global existence of the solution to the system (1.7), the point is to control the quantity $\|r^{-1}u^r\|_{L^1_t(L^\infty)}$. Indeed, compared with the case in the 3-D axisymmetric Euler equations, the vertical dissipation provides the bound of $\|r^{-1}u^r\|_{L^1_t(L^\infty)}$ by $\|\partial_z\frac{\omega}{r}\|_{L^1_t(L^{\frac{3}{2},1})}$ according to Proposition 2.2. Toward this, we introduce the unknowns $(\Omega, \Gamma) := \big(\frac{\omega}{r}, \frac{b}{r}\big)$ satisfying

$$\begin{cases} \partial_t\Omega + (u\cdot\nabla)\Omega - \partial_z^2\Omega = -\partial_z(\Gamma^2), \\ \partial_t\Gamma + (u\cdot\nabla)\Gamma - \partial_z^2\Gamma = 0. \end{cases} \tag{1.10}$$

The energy method applied to (1.10) may give necessary *a priori* estimates for the proof of the global existence of the solution to the system (1.7) with the initial data (1.8). Nevertheless, it's subtle to get the uniqueness of the solution to (1.7) since the above estimates is not sufficient to ensure the control of the quantity $\|\nabla u\|_{L^1_t(L^\infty)}$. Our strategy for proving the uniqueness lies in estimating the system (see (4.3) in Section 4) satisfied by the differences between two solutions with the same initial data. Due to the presence of the vertical dissipations in (4.3), we need only to bound the quantities $\mathcal{F}_i(t)$ with $i = 1, ..., 5$ in (4.4)-(4.12) below. Toward this, we can adopt the energy method to get the bounds of these quantities under the assumptions (1.9).

The rest of the paper is organized as follows. In Section 2, we recall some properties of the Lorentz spaces and basic lemmas on axisymmetric functions. Section 3 is devoted to some a priori estimates for the system (1.5). Finally, we present the proof of Theorem 1.1 in Section 4.

Notations: We shall denote $\int_{\mathbb{R}^3}\cdot dx = 2\pi\int_0^\infty\int_{\mathbb{R}}\cdot rdrdz$. For $A \lesssim B$, what we mean is that there exists a universal constant C, which may vary from line to line, such that $A \le CB$. Given a Banach space X, we shall use $(a|b)$ to represent the $L^2(\mathbb{R}^3)$ inner product of a and b, and $\|\,(a,b)\,\|_X = \|a\|_X + \|b\|_X$. The notation C_p is a positive constant depending on p.

2. Preliminaries

Before to introduce the definition of the Lorentz space, we begin by recalling the rearrangement of a function. For a measurable function f we define its non-increasing rearrangement by $f^* : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f^*(\lambda) \stackrel{\text{def}}{=} \inf\Big\{s \geq 0;\ \big|\{x|\,|f(x)| > s\}\big| \leq \lambda\Big\}.$$

Definition 2.1. *(Lorentz spaces, see [6]) Let f be a mesurable function and $1 \leq p,\, q \leq \infty$. Then f belongs to the Lorentz space $L^{p,q}$ if*

$$\|f\|_{L^{p,q}} \stackrel{def}{=} \begin{cases} \Big(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty & \text{if} \quad q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty & \text{if} \quad q = \infty. \end{cases}$$

Alternatively, we can define the Lorentz spaces by the real interpolation (see [6]), as the interpolation between Lebesgue spaces:

$$L^{p,q} \stackrel{\text{def}}{=} (L^{p_0}, L^{p_1})_{(\theta,q)},$$

with $1 \leq p_0 < p < p_1 \leq \infty$, $0 < \theta < 1$ satisfying $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $1 \leq q \leq \infty$, also $f \in L^{p,q}$ if the following quantity

$$\|f\|_{L^{p,q}} \stackrel{\text{def}}{=} \Big(\int_0^\infty \big(t^{-\theta} K(t,f)\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$

is finite with

$$K(f,t) \stackrel{\text{def}}{=} \inf_{f=f_0+f_1} \big\{\|f_0\|_{L^{p_0}} + t\|f_1\|_{L^{p_1}} \ \big|\ f_0 \in L^{p_0},\ f_1 \in L^{p_1}\big\}.$$

The Lorentz spaces verify the following properties (see [23] for more details) :

Proposition 2.1. *Let $f \in L^{p_1,q_1}$, $g \in L^{p_2,q_2}$ and $1 \leq p, q, p_j, q_j \leq \infty$ for $j = 1, 2$.*

- *If $1 < p < \infty$ and $1 \leq q \leq \infty$, then*
$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p,q}} \|g\|_{L^\infty}.$$
- *If $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*
$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$
- *If $1 < p < \infty$ and $1 \leq q \leq \infty$, then*
$$\|f * g\|_{L^{p,q}} \lesssim \|f\|_{L^{p,q}} \|g\|_{L^1}.$$
- *If $1 < p,\, p_1,\, p_2 < \infty$, $\frac{1}{p} + 1 = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, then*
$$\|f * g\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}},$$
for $p = \infty$, and $\frac{1}{q_1} + \frac{1}{q_2} = 1$, then
$$\|f * g\|_{L^\infty} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$
- *For $1 \leq p \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, we have*
$$L^{p,q_1} \hookrightarrow L^{p,q_2} \qquad \text{and} \qquad L^{p,p} = L^p.$$

Let us recall also the interpolation inequality (see [11]) which allows us to obtain some embeddings of spaces.

Lemma 2.1. *Let $p_0,\, p_1,\, p,\, q$ in $[1, +\infty]$ and $0 < \theta < 1$.*

- *If $q \leq p$, then*
$$\big[L^p(L^{p_0}), L^p(L^{p_1})\big]_{(\theta,q)} \hookrightarrow L^p\big([L^{p_0}, L^{p_1}]_{(\theta,q)}\big).$$

- *If $p \le q$, then*
$$L^p\big([L^{p_0},L^{p_1}]_{(\theta,q)}\big) \hookrightarrow \big[L^p(L^{p_0}),L^p(L^{p_1})\big]_{(\theta,q)}.$$

Recall also the definition of Lebesgue anisotropic spaces. Denote the space $L^p_v(\mathbb{R};\, L^q(\mathbb{R}^2))$ by $L^p_v(L^q_h)$ with the norm
$$\|f\|_{L^p_v(L^q_h)} \overset{\text{def}}{=} \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^2}|f(x,y,z)|^q\,dxdy\Big)^{\frac{p}{q}}dz\Big)^{\frac{1}{p}}.$$
Similarly, we denote by $L^q_h(L^p_v)$ the space $L^q(\mathbb{R}^2;\, L^p(\mathbb{R}))$, with the norm
$$\|f\|_{L^q_h(L^p_v)} \overset{\text{def}}{=} \Big(\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}}|f(x,y,z)|^p\,dz\Big)^{\frac{q}{p}}dxdy\Big)^{\frac{1}{q}}.$$

Lemma 2.2. *(See Lemma 3.1 in* [3]*) Let $1 \le p \le 2$ and $f \in L^p(\mathbb{R}^n)$ such that $\partial_i|f|^{\frac{p}{2}} \in L^2(\mathbb{R}^n)$. Then*
$$\|\partial_i f\|_{L^p} \lesssim \|\partial_i|f|^{\frac{p}{2}}\|_{L^2}\|f\|_{L^p}^{\frac{2-p}{2}}. \tag{2.1}$$

Thanks to Proposition 3.1 in [3], we readily get the following proposition (up to a slight modification).

Proposition 2.2. *Let u and b be axisymmetric solenoidal vector-field and scalar function respectively with vorticity $\omega = \omega^\theta e_\theta$, which solves the system* (1.7)*. Let $(p,q,\lambda) \in [1,\infty]^3$, then we have*
$$u^r = \omega^\theta = b = 0 \quad \text{on the axis} \quad r = 0,$$
and the following inequalities :

- *If $\frac{3}{2} \le p < \infty$ such that $\frac{1}{q} = \frac{1}{3} + \frac{1}{p}$, then*
$$\|u\|_{L^{p,\lambda}} \lesssim \|\omega\|_{L^{q,\lambda}}, \quad \|\frac{u^r}{r}\|_{L^{p,\lambda}} \lesssim \|\frac{\omega}{r}\|_{L^{q,\lambda}}, \quad \|\partial_z u^r\|_{L^{p,\lambda}} \lesssim \|\partial_z\omega\|_{L^{q,\lambda}},$$
$$\|\partial_z u^z\|_{L^{p,\lambda}} \lesssim \|\partial_z\omega\|_{L^{q,\lambda}}, \quad \|\partial_z u^z\|_{L^{p,\lambda}} + \|\partial_r u^z\|_{L^{p,\lambda}} \lesssim \|\partial_r\omega\|_{L^{q,\lambda}} + \|\frac{\omega}{r}\|_{L^{q,\lambda}}.$$
- *If $3 \le p < \infty$ such that $\frac{1}{q} = \frac{2}{3} + \frac{1}{p}$, then*
$$\|u^r\|_{L^{p,\lambda}} \lesssim \|\partial_z\omega\|_{L^{q,\lambda}}, \quad \|\frac{u^r}{r}\|_{L^{p,\lambda}} \lesssim \|\partial_z\frac{\omega}{r}\|_{L^{q,\lambda}}, \quad \|u^z\|_{L^{p,\lambda}} \lesssim \|\partial_r\omega\|_{L^{q,\lambda}} + \|\frac{\omega}{r}\|_{L^{q,\lambda}},$$
$$\|\partial_z u^z\|_{L^{p,\lambda}} \lesssim \|\partial_z\partial_r\omega\|_{L^{q,\lambda}} + \|\partial_z\frac{\omega}{r}\|_{L^{q,\lambda}}, \quad \|\partial_r u^r\|_{L^{p,\lambda}} \lesssim \|\partial_z\partial_r\omega\|_{L^{q,\lambda}} + \|\partial_z\frac{\omega}{r}\|_{L^{q,\lambda}}.$$
- *In the limiting case $p = \infty$, there hold*
$$\|u\|_{L^\infty} \lesssim \|\omega\|_{L^{3,1}}, \quad \|u^r\|_{L^\infty} \lesssim \|\partial_z\omega\|_{L^{\frac{3}{2},1}}, \quad \|\frac{u^r}{r}\|_{L^\infty} \lesssim \|\partial_z\frac{\omega}{r}\|_{L^{\frac{3}{2},1}},$$
$$\|u^z\|_{L^\infty} \lesssim \|\partial_r\omega\|_{L^{\frac{3}{2},1}} + \|\frac{\omega}{r}\|_{L^{\frac{3}{2},1}}, \quad \|\partial_z u^z\|_{L^\infty} \lesssim \|\partial_z\partial_r\omega\|_{L^{\frac{3}{2},1}} + \|\partial_z\frac{\omega}{r}\|_{L^{\frac{3}{2},1}},$$
$$\|\partial_r u^r\|_{L^\infty} \lesssim \|\partial_z\partial_r\omega\|_{L^{\frac{3}{2},1}} + \|\partial_z\frac{\omega}{r}\|_{L^{\frac{3}{2},1}}.$$

The proposition given below can be found in [22], which we will use in the proof of higher order estimates of (u,b).

Proposition 2.3. *(*[22]*) Let u be a free divergence axisymmetric vector-field without swirl and $\omega = \nabla \times u$. Then there hold*
$$\frac{u^r}{r} = \partial_z\Delta^{-1}(\frac{\omega}{r}) - 2\frac{\partial_r}{r}\Delta^{-1}\partial_z\Delta^{-1}(\frac{\omega}{r})$$
and
$$\|\partial_z(\frac{u^r}{r})\|_{L^p} \le C\|\frac{\omega}{r}\|_{L^p}, \quad 1 < p < \infty.$$

3. A prior estimates

Proposition 3.1. *Assume that* $1 < p < \infty$, $(\Omega_0, \Gamma_0) \in L^p \times L^{2p}$, *u and b are regular axisymmetric such that* $\mathrm{div}\, u = 0$. *Let* $\Omega \overset{def}{=} \frac{\omega}{r} \in L^\infty_t(L^p)$ *and* $\Gamma \overset{def}{=} \frac{b}{r} \in L^\infty_t(L^p)$ *be regular solutions of the system* (1.10). *Then there are*

$$\|\Gamma(t)\|_{L^p} + C_p \|\partial_z |\Gamma|^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2_t(L^2)} \le \|\Gamma_0\|_{L^p}, \tag{3.1}$$

and

$$\|\Omega(t)\|_{L^p} + C_p \|\partial_z |\Omega|^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2_t(L^2)} \le C\big(\|\Omega_0\|_{L^p} + \sqrt{t}\|\Gamma_0^2\|_{L^p}\big). \tag{3.2}$$

Proof. Let's first control Γ in Lebesgue spaces. For $1 < p < \infty$, multiplying the second equation of (1.10) by $|\Gamma|^{p-1}\mathrm{sign}\,\Gamma$, and then integrating by parts, we obtain from $\mathrm{div}\, u = 0$ that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma\|^p_{L^p} + \frac{4(p-1)}{p^2}\|\partial_z |\Gamma|^{\frac{p}{2}}\|^2_{L^2} = 0, \tag{3.3}$$

which yields (3.1).

In order to control Ω in Lebesgue spaces, we will split it into two cases: $1 < p \le 2$ and $2 \le p < \infty$.

Case 1 : $1 < p \le 2$. Taking the L^2 inner product of the second equation (1.10) with $|\Omega|^{p-1}\,\mathrm{sign}(\Omega)$, we find

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega\|^p_{L^p} + \frac{4(p-1)}{p^2}\|\partial_z |\Omega|^{\frac{p}{2}}\|^2_{L^2} \le \int_{\mathbb{R}^3} -\partial_z \Gamma^2 |\Omega|^{p-1}\mathrm{sign}\,\Omega \mathrm{d}x, \tag{3.4}$$

which implies

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega\|^p_{L^p} + \frac{4(p-1)}{p^2}\|\partial_z |\Omega|^{\frac{p}{2}}\|^2_{L^2} \le \|\partial_z \Gamma^2\|_{L^p}\|\Omega\|^{p-1}_{L^p}. \tag{3.5}$$

Hence, there holds

$$\|\Omega(t)\|_{L^p} + C_p \|\partial_z |\Omega|^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2_t(L^2)} \le \|\Omega_0\|_{L^p} + \int_0^t \|\partial_z \Gamma^2\|_{L^p}\mathrm{d}\tau. \tag{3.6}$$

In order to close the above inequality, we may obtain the equation of Γ^2 from the Γ-equation in (1.10) that

$$\partial_t \Gamma^2 + (u\cdot\nabla)\Gamma^2 - \partial_z^2 \Gamma^2 = -2(\partial_z \Gamma)^2. \tag{3.7}$$

Similar to the argument in (3.4), we have

$$\|\Gamma^2(t)\|^p_{L^p} + C_p \|\partial_z |\Gamma^2|^{\frac{p}{2}}\|^2_{L^2_t(L^2)} \le \|\Gamma_0^2\|^p_{L^p}. \tag{3.8}$$

Thanks to (2.1), we have

$$\int_0^t \|\partial_z \Gamma^2\|^2_{L^p}\, d\tau \le C\|\partial_z |\Gamma^2|^{\frac{p}{2}}\|^2_{L^2_t(L^2)} \|\Gamma^2\|^{2-p}_{L^\infty_t(L^p)} \le C\|\Gamma_0^2\|^2_{L^p}.$$

and then, we get, for $1 < p \le 2$,

$$\|\Gamma^2(t)\|_{L^p} + C_p \|\partial_z \Gamma^2\|_{L^2_t(L^p)} \le C\|\Gamma_0^2\|_{L^p}, \tag{3.9}$$

and

$$\|\partial_z \Gamma^2\|_{L^1_t(L^p)} \le \sqrt{t}\|\partial_z \Gamma^2\|_{L^2_t(L^p)} \le C\sqrt{t}\|\Gamma_0^2\|_{L^p}. \tag{3.10}$$

Inserting (3.10) into (3.6) implies (3.2).

Case 2: $2 \leq p < +\infty$**.** Thanks to (3.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega\|_{L^p}^p + \frac{4(p-1)}{p}\|\partial_z|\Omega|^{\frac{p}{2}}\|_{L^2}^2 \leq \frac{2p-2}{p}\Big|\int_{\mathbb{R}^3}\Gamma^2(\partial_z|\Omega|^{\frac{p}{2}})|\Omega|^{\frac{p-2}{2}}\mathrm{d}x\Big|, \tag{3.11}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega\|_{L^p}^p + \frac{4(p-1)}{p}\|\partial_z|\Omega|^{\frac{p}{2}}\|_{L^2}^2 \leq C_p\|\partial_z|\Omega|^{\frac{p}{2}}\|_{L^2}\|\Gamma^2\|_{L^p}\|\Omega\|_{L^p}^{\frac{p-2}{2}}. \tag{3.12}$$

Thence, Young's inequality implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega\|_{L^p}^p + \frac{2(p-1)}{p}\|\partial_z|\Omega|^{\frac{p}{2}}\|_{L^2}^2 \leq C\|\Gamma^2\|_{L^p}^2\|\Omega\|_{L^p}^{p-2}. \tag{3.13}$$

Combining (3.13) with (3.8), one obtains (3.2).

Therefore, we finish the proof of Proposition 3.1. □

Remark 3.1. *For $1 < p \leq 2$, thanks to* (2.1)*, we have*

$$\|\partial_z\Gamma\|_{L^p} \lesssim \|\partial_z|\Gamma|^{\frac{p}{2}}\|_{L^2}\|\Gamma\|_{L^p}^{\frac{2-p}{2}}, \quad \|\partial_z\Omega\|_{L^p} \lesssim \|\partial_z|\Omega|^{\frac{p}{2}}\|_{L^2}\|\Omega\|_{L^p}^{\frac{2-p}{2}},$$

which along with (3.1) *and* (3.2) *implies that*

$$\begin{aligned} &\|\Gamma(t)\|_{L^p} + \|\partial_z\Gamma\|_{L^2_t(L^p)} \leq C\|\Gamma_0\|_{L^p},\\ &\|\Gamma^2(t)\|_{L^p} + \|\partial_z\Gamma^2\|_{L^2_t(L^p)} \leq C\|\Gamma_0^2\|_{L^p},\\ &\|\Omega(t)\|_{L^p} + \|\partial_z\Omega\|_{L^2_t(L^p)} \leq C\big(\|\Omega_0\|_{L^p} + \sqrt{t}\|\Gamma_0^2\|_{L^p}\big). \end{aligned} \tag{3.14}$$

Remark 3.2. *We denote by $\mathcal{T}$ and $\mathcal{S}$ the following linear operators:*

$$\begin{aligned} \mathcal{T}: \quad & L^p \longrightarrow L^p & \qquad \mathcal{S}: \quad & L^p \longrightarrow L^2_t(L^p)\\ & \Omega_0 \longmapsto \Omega & & \Omega_0 \longmapsto \partial_z\Omega, \end{aligned}$$

with Ω solution of the system (1.10)*. By definition, we know that $\mathcal{T}$ and $\mathcal{S}$ are linear operators, then thanks to Propositions 2.1 and 3.1, Lemmas 2.1 and 2.2, and Remark 3.1, we obtain for $1 < p \leq 2$, $1 \leq q \leq p$,*

$$\begin{aligned} &\|\Omega(t)\|_{L^{p,q}} + \Big\|\partial_z\Omega\Big\|_{L^2_t(L^{p,q})} \leq C\Big(\|\Omega_0\|_{L^{p,q}} + \sqrt{t}\|\Gamma_0\|_{L^{2p,2q}}^2\Big),\\ &\|\Gamma(t)\|_{L^{p,q}} + \|\partial_z\Gamma\|_{L^2_t(L^{p,q})} \leq C\|\Gamma_0\|_{L^{p,q}}, \quad \|\Gamma^2(t)\|_{L^{p,q}} + \|\partial_z\Gamma^2\|_{L^2_t(L^{p,q})} \leq C\|\Gamma_0^2\|_{L^{p,q}}. \end{aligned} \tag{3.15}$$

While for $2 < p < \infty$, $1 \leq q \leq p$, we have

$$\|\Omega(t)\|_{L^{p,q}} \leq C\Big(\|\Omega_0\|_{L^{p,q}} + \sqrt{t}\|\Gamma_0\|_{L^{2p,2q}}^2\Big) \quad \text{and} \quad \|\Gamma(t)\|_{L^{p,q}} \leq \|\Gamma_0\|_{L^{p,q}}. \tag{3.16}$$

Corollary 3.1. *Assume that $(\Omega_0, \Gamma_0) \in L^{\frac{3}{2},1} \times L^{3,2}$ and u a regular axisymmetric vector field such that $\operatorname{div} u = 0$. Let $\Omega \overset{def}{=} \frac{\omega}{r} \in L^\infty_t(L^{\frac{3}{2},1})$ and $\Gamma \overset{def}{=} \frac{b}{r} \in L^\infty_t(L^{3,2})$ be a solution of system* (1.10)*. Then there are*

$$\begin{aligned} &\|\Gamma(t)\|_{L^{3,2}} + \|\partial_z\Gamma^2\|_{L^2_t(L^{\frac{3}{2},1})}^{\frac{1}{2}} \leq \|\Gamma_0\|_{L^{3,2}},\\ &\|\Omega(t)\|_{L^{\frac{3}{2},1}} + \|\partial_z\Omega\|_{L^2_t(L^{\frac{3}{2},1})} \leq C\big(\|\Omega_0\|_{L^{\frac{3}{2},1}} + \sqrt{t}\|\Gamma_0\|_{L^{3,2}}^2\big). \end{aligned} \tag{3.17}$$

In particular, we have for all $t \geq 0$,

$$\int_0^t \|\frac{u^r}{r}\|_{L^\infty}\mathrm{d}\tau \lesssim \int_0^t \|\partial_z\frac{\omega}{r}\|_{L^{\frac{3}{2},1}}\mathrm{d}\tau \leq C\sqrt{t}\big(\|\Omega_0\|_{L^{\frac{3}{2},1}} + \sqrt{t}\|\Gamma_0\|_{L^{3,2}}^2\big). \tag{3.18}$$

Proposition 3.2. *Let* $1 < p < \infty$, $\frac{\omega_0}{r} \in L^{\frac{3}{2},1}$, $\frac{b_0}{r} \in L^{3,2}$, $\omega_0, b_0, \frac{b_0^2}{r} \in L^p$. *Assume that* $\omega \in L^\infty_t(L^p)$ *and* $b \in L^\infty_t(L^p)$ *be a solution of the equations* (1.7)*. Then there hold*

$$\|\omega(t)\|_{L^p} + \Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^{\frac{2}{p}}_{L^2_t(L^2)} \leq C\big(\|\omega_0\|_{L^p} + \sqrt{t}\,\|\frac{b_0^2}{r}\|_{L^p}\big)e^{CA_0(t)}, \tag{3.19}$$

$$\|\frac{b^2}{r}(t)\|_{L^p} + \Big\|\partial_z|\frac{b^2}{r}|^{\frac{p}{2}}\Big\|^{\frac{2}{p}}_{L^2_t(L^2)} \leq C\|\frac{b_0^2}{r}\|_{L^p}e^{CA_0(t)}, \tag{3.20}$$

and

$$\|b(t)\|_{L^p} + \|\partial_z|b|^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2_t(L^2)} \leq C\|b_0\|_{L^p}e^{CA_0(t)}, \tag{3.21}$$

where

$$A_0(t) \overset{def}{=} \sqrt{t}\big(\|\Omega_0\|_{L^{\frac{3}{2},1}} + \sqrt{t}\,\|\Gamma_0\|^2_{L^{3,2}}\big).$$

Proof. Due to the second equation in (1.7), we find

$$\partial_t(\frac{b^2}{r}) + (u\cdot\nabla)(\frac{b^2}{r}) - \partial_z^2(\frac{b^2}{r}) = -\frac{2}{r}(\partial_z b)^2 + \frac{u^r}{r}(\frac{b^2}{r}). \tag{3.22}$$

Hence, for any $1 < p < +\infty$, we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\frac{b^2}{r}\|^p_{L^p} + \frac{4(p-1)}{p^2}\Big\|\partial_z|\frac{b^2}{r}|^{\frac{p}{2}}\Big\|^2_{L^2} = -\int_{\mathbb{R}^3}\frac{2}{r}(\partial_z b)^2(\frac{b^2}{r})^{p-1}\,\mathrm{d}x + \int_{\mathbb{R}^3}\frac{u^r}{r}(\frac{b^2}{r})^p\,\mathrm{d}x, \tag{3.23}$$

which implies

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\frac{b^2}{r}\|^p_{L^p} + \frac{4(p-1)}{p^2}\Big\|\partial_z|\frac{b^2}{r}|^{\frac{p}{2}}\Big\|^2_{L^2} \leq \|\frac{u^r}{r}\|_{L^\infty}\|\frac{b^2}{r}\|^p_{L^p}. \tag{3.24}$$

Gronwall's inequality along with (3.18) leads to

$$\sup_{\tau\in[0,t]}\|\frac{b^2}{r}(\tau)\|^p_{L^p} + \Big\|\partial_z|\frac{b^2}{r}|^{\frac{p}{2}}\Big\|^2_{L^2_t(L^2)} \leq C\|\frac{b_0^2}{r}\|^p_{L^p}\exp\{C\int_0^t\|\frac{u^r}{r}\|_{L^\infty}(\tau)\,\mathrm{d}\tau\} \leq \|\frac{b_0^2}{r}\|^p_{L^p}e^{CA_0(t)}, \tag{3.25}$$

and then

$$\sup_{\tau\in[0,t]}\|\frac{b^2}{r}(\tau)\|^p_{L^p} + \Big\|\partial_z|\frac{b^2}{r}|^{\frac{p}{2}}\Big\|^2_{L^2_t(L^2)} \leq C\|\frac{b_0^2}{r}\|^p_{L^p}e^{CA_0(t)}. \tag{3.26}$$

Similarly, from the b equation of (1.7), we have

$$\sup_{\tau\in[0,t]}\|b(\tau)\|_{L^p} + \Big\|\partial_z|b|^{\frac{p}{2}}\Big\|^{\frac{2}{p}}_{L^2} \leq C\|b_0\|_{L^p}e^{CA_0(t)}. \tag{3.27}$$

Case 1: $2 \leq p < +\infty$**.** Taking the L^2 inner product of the vorticity equation in (1.7) with $|\omega|^{p-1}\mathrm{sign}(\omega)$, we obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^p_{L^p} + \frac{4(p-1)}{p^2}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^2_{L^2} &= \int_{\mathbb{R}^3}\frac{u^r}{r}|\omega|^p\,\mathrm{d}x - \int_{\mathbb{R}^3}\partial_z(\frac{b^2}{r})|\omega|^{p-1}\mathrm{sign}(\omega)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}\frac{u^r}{r}|\omega|^p\,\mathrm{d}x + \int_{\mathbb{R}^3}(\frac{b^2}{r})\partial_z(|\omega|^{p-1}\mathrm{sign}(\omega))\,\mathrm{d}x.\end{aligned} \tag{3.28}$$

Hence, using Hölder's and Young's inequalities, one has

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^p_{L^p} + \frac{4(p-1)}{p^2}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^2_{L^2} &\leq \|\frac{u^r}{r}\|_{L^\infty}\|\omega\|^p_{L^p} + (p-1)\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L^2}\Big[\int_{\mathbb{R}^3}\frac{b^4}{r^2}|\omega|^{p-2}\mathrm{d}x\Big]^{\frac{1}{2}}\\ &\leq \eta\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|^2_{L^2} + \|\frac{u^r}{r}\|_{L^\infty}\|\omega\|^p_{L^p} + C_\eta\|\frac{b^2}{r}\|^2_{L^p}\|\omega\|^{p-2}_{L^p}\end{aligned} \tag{3.29}$$

for any positive constant η. Hence, taking $\eta = \frac{2(p-1)}{p^2}$, we get

$$\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{2(p-1)}{p}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L^2}^2 \leq C\|\frac{u^r}{r}\|_{L^\infty}\|\omega\|_{L^p}^p + C\|\frac{b^2}{r}\|_{L^p}^2\|\omega\|_{L^p}^{p-2}. \tag{3.30}$$

Gronwall's inequality implies

$$\begin{aligned}\sup_{\tau\in[0,t]}\|\omega(\tau)\|_{L^p}^p + \Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L_t^2(L^2)}^2 &\leq C(\|\omega_0\|_{L^p}^p + \int_0^t \|\frac{b^2}{r}\|_{L^p}^2\|\omega\|_{L^p}^{p-2}\,d\tau)e^{C\int_0^t\|\frac{u^r}{r}(\tau)\|_{L^\infty}\,d\tau}\\ &\leq C(\|\omega_0\|_{L^p}^p + t\,\|\frac{b^2}{r}\|_{L_t^\infty(L^p)}^2\|\omega\|_{L_t^\infty(L^p)}^{p-2})e^{CA_0(t)},\end{aligned} \tag{3.31}$$

which along with (3.25) implies

$$\|\omega(t)\|_{L^p} + \Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L_t^2(L^2)}^{\frac{2}{p}} \leq C\big(\|\omega_0\|_{L^p} + \|\frac{b_0^2}{r}\|_{L^p}\sqrt{t}\big)e^{CA_0(t)}. \tag{3.32}$$

Case 2: $1<p<2$**.** The energy estimates infer that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p}^p + \frac{4(p-1)}{p^2}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L^2}^2 &= \int_{\mathbb{R}^3}\frac{u^r}{r}|\omega|^p\,dx - \int_{\mathbb{R}^3}\partial_z(\frac{b^2}{r})|\omega|^{p-1}\mathrm{sign}(\omega)\,dx\\ &\leq \|\frac{u^r}{r}\|_{L^\infty}\|\omega\|_{L^p}^p + \|\partial_z(\frac{b^2}{r})\|_{L^p}\|\omega\|_{L^p}^{p-1}.\end{aligned} \tag{3.33}$$

which along with (2.1) gives rise to

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^p}^p + \frac{4(p-1)}{p^2}\Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L^2}^2 \leq \|\frac{u^r}{r}\|_{L^\infty}\|\omega\|_{L^p}^p + C\|\partial_z(\frac{b^2}{r})^{\frac{p}{2}}\|_{L^2}\|\frac{b^2}{r}\|_{L^p}^{\frac{2-p}{2}}\|\omega\|_{L^p}^{p-1}. \tag{3.34}$$

Thanks to Gronwall's inequality, we deduce

$$\sup_{\tau\in[0,t]}\|\omega(\tau)\|_{L^p} + \Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L_t^2(L^2)}^{\frac{2}{p}} \leq C\big(\|\omega_0\|_{L^p} + \int_0^t\|\partial_z(\frac{b^2}{r})^{\frac{p}{2}}\|_{L^2}\|\frac{b^2}{r}\|_{L^p}^{\frac{2-p}{2}}\,\mathrm{d}\tau\big)e^{CA_0(t)}, \tag{3.35}$$

which follows

$$\sup_{\tau\in[0,t]}\|\omega(\tau)\|_{L^p} + \Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L_t^2(L^2)}^{\frac{2}{p}} \leq C\big(\|\omega_0\|_{L^p} + \sqrt{t}\|\partial_z(\frac{b^2}{r})^{\frac{p}{2}}\|_{L_t^2(L^2)}\|\frac{b^2}{r}\|_{L_t^\infty(L^p)}^{\frac{2-p}{2}}\big)e^{CA_0(t)}. \tag{3.36}$$

Therefore, due to (3.26), we find

$$\sup_{\tau\in[0,t]}\|\omega(\tau)\|_{L^p} + \Big\|\partial_z|\omega|^{\frac{p}{2}}\Big\|_{L_t^2(L^2)}^{\frac{2}{p}} \leq C\big(\|\omega_0\|_{L^p} + \sqrt{t}\,\|\frac{b_0^2}{r}\|_{L^p}\big)e^{CA_0(t)}. \tag{3.37}$$

This completes the proof of the proposition. ☐

Thanks to Proposition 3.2 and Lemma 2.1, we have the following results.

Corollary 3.2. *Let the initial data* (ω_0, b_0) *satisfy*

$$\omega_0\in L^{p,q},\quad \frac{\omega_0}{r}\in L^{\frac{3}{2},1},\quad b_0\in L^{3,2}\cap L^{p,q},\quad \text{and}\quad \frac{b_0}{r}\in L^{2p,2q}\cap L^{3,2}.$$

Assume that $1<p<\infty$, $1\leq q\leq p$, $\omega\in L_t^\infty(L^{p,q})$ *and* $b\in L_t^\infty(L^{p,q})$ *be a solution of equation* (1.7)*. Then there are*

- *if* $1<p\leq 2$, $1\leq q\leq p$, *then*
  - $\|\omega(t)\|_{L^{p,q}} + \|\partial_z\omega\|_{L_t^2(L^{p,q})} \leq C\big(\|\omega_0\|_{L^{p,q}} + \sqrt{t}\,\|\frac{b_0^2}{r}\|_{L^{p,q}}\big)e^{CA_0(t)}$,
  - $\|b(t)\|_{L^{p,q}} + \|\partial_z b\|_{L_t^2(L^{p,q})} \leq C\|b_0\|_{L^{p,q}}e^{CA_0(t)}$,
  - $\|\frac{b^2}{r}(t)\|_{L^{p,q}} + \|\partial_z\frac{b^2}{r}\|_{L_t^2(L^{p,q})} \leq C\|\frac{b_0^2}{r}\|_{L^{p,q}}e^{CA_0(t)}$.
- *if* $2<p<\infty$, $1\leq q\leq p$, *then*

– $\|\omega(t)\|_{L^{p,q}} \le C\big(\|\omega_0\|_{L^{p,q}} + \sqrt{t}\,\|\frac{b_0^2}{r}\|_{L^{p,q}}\big)e^{CA_0(t)}$,
– $\|b(t)\|_{L^{p,q}} \le C\|b_0\|_{L^{p,q}}e^{CA_0(t)}$,
– $\|\frac{b^2}{r}(t)\|_{L^{p,q}} \le C\|\frac{b_0^2}{r}\|_{L^{p,q}}e^{CA_0(t)}$.

In particular, if ω_0, $\frac{\omega_0}{r} \in L^{\frac32,1}$, $b_0 \in L^{3,2}$, *and* $r^{-1}b_0 \in L^{\frac32,1} \cap L^{3,2}$, *we have*

$$\begin{aligned}
&\|\omega\|_{L^\infty_t(L^{\frac32,1})} + \|\partial_z\omega\|_{L^2_t(L^{\frac32,1})} \le C\big(\|\omega_0\|_{L^{\frac32,1}} + \sqrt{t}\,\|\frac{b_0^2}{r}\|_{L^{\frac32,1}}\big)e^{CA_0(t)},\\
&\|r^{-1}\omega\|_{L^\infty_t(L^{\frac32,1})} + \|r^{-1}\partial_z\omega\|_{L^2_t(L^{\frac32,1})} \le C\|r^{-1}\omega_0\|_{L^{\frac32,1}} + \sqrt{t}\|r^{-1}b_0\|^2_{L^{3,2}},\\
&\|\frac{b}{r}\|_{L^\infty_t(L^{\frac32,1})} + \|\partial_z\frac{b}{r}\|_{L^2_t(L^{\frac32,1})} \le C\|\frac{b_0}{r}\|_{L^{\frac32,1}},\\
&\|\frac{b^2}{r}\|_{L^\infty_t(L^{\frac32,1})} + \|\partial_z\frac{b^2}{r}\|_{L^2_t(L^{\frac32,1})} \le C\|\frac{b_0^2}{r}\|_{L^{\frac32,1}}e^{CA_0(t)},\\
&\|\frac{b}{r}\|_{L^\infty_t(L^{3,2})} \le C\|\frac{b_0}{r}\|_{L^{3,2}}, \quad \|b\|_{L^\infty_t(L^{3,2})} \le C\|b_0\|_{L^{3,2}}e^{CA_0(t)}.
\end{aligned}\tag{3.38}$$

Below we give more higher-order estimates which will be used in the proof of uniqueness.

Proposition 3.3. *Assume that the initial data* (ω_0, b_0) *satisfies*

$$\omega_0 \in L^{3,1} \cap L^{\frac32,1}, \quad r^{-1}\omega_0 \in L^{\frac32,1}, \quad b_0 \in L^2 \cap L^{3,2}, \quad r^{-1}b_0 \in L^{3,2} \cap \dot H^1.$$

Let $\frac{b}{r}$ *a solution of the second equation of the system* (1.10). *Then*

$$\|\nabla\frac{b}{r}(t)\|^2_{L^2} + \|\partial_z\nabla\frac{b}{r}\|^2_{L^2_t(L^2)} \le C\|\nabla\frac{b_0}{r}\|^2_{L^2}\exp\{CA_0(t) + CA_1(t)e^{CA_0(t))}\},\tag{3.39}$$

where

$$A_1(t) \triangleq t\|\omega_0\|^2_{L^{3,1}} + t^2\|\nabla\frac{b_0}{r}\|^2_{L^2}\|b_0\|^2_{L^2} + \|\omega_0\|^2_{L^{\frac32,1}} + t\,\|b_0\|^2_{L^{3,2}}\|\frac{b_0}{r}\|^2_{L^{3,2}}.$$

Proof. Multiply both sides of the second of (1.10) by $-\Delta\frac{b}{r}$, then integrating by part yields

$$\begin{aligned}
&\|\nabla\frac{b}{r}\|^2_{L^2}(t) + \|\partial_z\nabla\frac{b}{r}\|^2_{L^2_t(L^2)} = 2\pi\int_{\mathbb{R}^2_+}(u^r\partial_r\frac{b}{r} + u^z\partial_z\frac{b}{r})(\frac1r\partial_r(r\partial_r\frac{b}{r}) + \partial_z^2\frac{b}{r})rdrdz\\
&= 2\pi\int_{\mathbb{R}^2_+}u^r\partial_r\frac{b}{r}\partial_r(r\partial_r\frac{b}{r})drdz + 2\pi\int_{\mathbb{R}^2_+}u^z\partial_z\frac{b}{r}\partial_r(r\partial_r\frac{b}{r})drdz\\
&\quad + 2\pi\int_{\mathbb{R}^2_+}(u^r\partial_r\frac{b}{r} + u^z\partial_z\frac{b}{r})\partial_z^2\frac{b}{r}rdrdz \triangleq I_1 + I_2 + I_3.
\end{aligned}\tag{3.40}$$

Note that $\partial_r u^r = -\frac{u^r}{r} - \partial_z u^z$, so we find

$$\begin{aligned}
I_1 &= \pi\int_{\mathbb{R}^2_+}\frac{u^r}{r}\partial_r(r\partial_r\frac{b}{r})^2drdz = \pi\int_{\mathbb{R}^2_+}\frac{u^r}{r^2}r^2(\partial_r\frac{b}{r})^2drdz - \pi\int_{\mathbb{R}^2_+}\frac1r\partial_r u^r r^2(\partial_r\frac{b}{r})^2drdz\\
&= 2\pi\int_{\mathbb{R}^2_+}\frac{u^r}{r}(\partial_r\frac{b}{r})^2rdrdz + \pi\int_{\mathbb{R}^2_+}\partial_z u^z(\partial_r\frac{b}{r})^2rdrdz\\
&= 2\pi\int_{\mathbb{R}^2_+}\frac{u^r}{r}(\partial_r\frac{b}{r})^2rdrdz - 2\pi\int_{\mathbb{R}^2_+}u^z(\partial_r\frac{b}{r})\partial_z\partial_r\frac{b}{r}rdrdz,
\end{aligned}$$

which implies

$$|I_1| \lesssim \|\frac{u^r}{r}\|_{L^\infty}\|\partial_r\frac{b}{r}\|^2_{L^2} + \|u\|_{L^\infty}\|\partial_z\partial_r\frac{b}{r}\|_{L^2}\|\partial_r\frac{b}{r}\|_{L^2}.$$

By virtue of $\partial_r u^z = \partial_z u^r - \omega$ and integration by parts, the bound of I_2 has

$$\begin{aligned} I_2 &= -2\pi \int_{\mathbb{R}^2_+} u^z \partial_z \partial_r \frac{b}{r} \partial_r \frac{b}{r} r dr dz - 2\pi \int_{\mathbb{R}^2_+} \partial_r u^z \partial_z \frac{b}{r} \partial_r \frac{b}{r} r dr dz \\ &= -\int_{\mathbb{R}^3} u^z \partial_z \partial_r \frac{b}{r} \partial_r \frac{b}{r} dx + \int_{\mathbb{R}^3} (\omega - \partial_z u^r) \partial_z \frac{b}{r} \partial_r \frac{b}{r} \mathrm{d}x, \end{aligned}$$

which along with the facts $\|\partial_z u^r\|_{L^3} \lesssim \|\partial_z \omega\|_{L^{\frac{3}{2}}}$ and $\|u\|_{L^\infty} \lesssim \|\omega\|_{L^{3,1}}$ gives rise to

$$\begin{aligned} |I_2| &\lesssim \|u^z\|_{L^\infty} \|\partial_z \partial_r \frac{b}{r}\|_{L^2} \|\partial_r \frac{b}{r}\|_{L^2} + (\|\omega\|_{L^3} + \|\partial_z u^r\|_{L^3}) \|\partial_z \frac{b}{r}\|_{L^6} \|\partial_r \frac{b}{r}\|_{L^2} \\ &\lesssim (\|\omega\|_{L^{3,1}} + \|\partial_z \omega\|_{L^{\frac{3}{2}}}) \|\partial_z \nabla \frac{b}{r}\|_{L^2} \|\nabla \frac{b}{r}\|_{L^2}. \end{aligned}$$

By Hölder's inequality, we have

$$|I_3| \lesssim \|u\|_{L^\infty} \|\nabla \frac{b}{r}\|_{L^2} \|\partial_z \nabla \frac{b}{r}\|_{L^2} \lesssim \|\omega\|_{L^{3,1}} \|\partial_z \nabla \frac{b}{r}\|_{L^2} \|\nabla \frac{b}{r}\|_{L^2}.$$

Substituting $I_1 - I_3$ estimates into (3.40), from the fact $\|u\|_{L^\infty} \lesssim \|\omega\|_{L^{3,1}}$, we obtain

$$\begin{aligned} &\|\nabla \frac{b}{r}(t)\|_{L^2}^2 + \|\partial_z \nabla \frac{b}{r}\|_{L^2_t(L^2)}^2 \\ &\lesssim \|\frac{u^r}{r}\|_{L^\infty} \|\nabla \frac{b}{r}\|_{L^2}^2 + \|\omega\|_{L^{3,1}} \|\nabla \frac{b}{r}\|_{L^2} \|\partial_z \nabla \frac{b}{r}\|_{L^2} + \|\partial_z \omega\|_{L^{\frac{3}{2}}} \|\partial_z \nabla \frac{b}{r}\|_{L^2} \|\nabla \frac{b}{r}\|_{L^2}, \end{aligned}$$

which along with Young's inequality implies

$$\|\nabla \frac{b}{r}(t)\|_{L^2}^2 + \|\partial_z \nabla \frac{b}{r}\|_{L^2_t(L^2)}^2 \leq C(\|\frac{u^r}{r}\|_{L^\infty} + \|\omega\|_{L^{3,1}}^2 + \|\partial_z \omega\|_{L^{\frac{3}{2}}}^2) \|\nabla \frac{b}{r}\|_{L^2}^2.$$

Hence, applying Gronwall's inequality gives rise to

$$\|\nabla \frac{b}{r}(t)\|_{L^2}^2 + \|\partial_z \nabla \frac{b}{r}\|_{L^2_t(L^2)}^2 \leq C \|\nabla \frac{b_0}{r}\|_{L^2}^2 \exp\{C \int_0^t (\|\frac{u^r}{r}\|_{L^\infty} + \|\omega\|_{L^{3,1}}^2 + \|\partial_z \omega\|_{L^{\frac{3}{2}}}^2) d\tau\}. \tag{3.41}$$

Thanks to Corollary 3.1, Proposition 3.2, and the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^{6,2}(\mathbb{R}^3)$ (see [25, 6, 28]), we know that

$$\begin{aligned} \|\frac{u^r}{r}\|_{L^\infty_t(L^\infty)} &\leq C(\|\Omega_0\|_{L^{\frac{3}{2},1}} + \sqrt{t} \|\Gamma_0\|_{L^{3,2}}^2), \\ \|\omega\|_{L^\infty_t(L^{3,1})} &\leq C\big(\|\omega_0\|_{L^{3,1}} + \sqrt{t} \|\frac{b_0}{r}\|_{L^{6,2}} \|b_0\|_{L^{6,2}}\big) e^{CA_0(t)} \\ &\leq C\big(\|\omega_0\|_{L^{3,1}} + \sqrt{t} \|\nabla \frac{b_0}{r}\|_{L^2} \|\nabla\, b_0\|_{L^2}\big) e^{CA_0(t)}, \\ \|\partial_z \omega\|_{L^2_t(L^{\frac{3}{2},1})}^2 &\leq C\big(\|\omega_0\|_{L^{\frac{3}{2},1}}^2 + t\, \|b_0\|_{L^{3,2}}^2 \|\frac{b_0}{r}\|_{L^{3,2}}^2\big) e^{CA_0(t)}. \end{aligned}$$

Substituting the above inequalities into (3.41), we get (3.39), which concludes the proof of Proposition 3.3. $\square$

Remark 3.3. *Thanks to Theorem 5.3.1. in* [6]*, we have the following interpolation inequality*

$$\|b_0\|_{L^{3,2}(\mathbb{R}^3)} \lesssim \|b_0\|_{L^{\frac{3}{2},2}(\mathbb{R}^3)}^{\frac{1}{3}} \|b_0\|_{L^{6,2}(\mathbb{R}^3)}^{\frac{2}{3}} \lesssim \|b_0\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{1}{3}} \|\nabla\, b_0\|_{L^2(\mathbb{R}^3)}^{\frac{2}{3}},$$

where we used the embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^{6,2}(\mathbb{R}^3)$ (see [25, 28]*), which implies that*

$$L^{\frac{3}{2}}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3) \subset L^{3,2}(\mathbb{R}^3).$$

Proposition 3.4. *Assume that the initial data* (ω_0, b_0) *satisfies*

$$\omega_0 \in L^{3,1}, \quad r^{-1}\omega_0 \in L^{\frac{3}{2},1}, \quad b_0 \in L^{3,2} \cap \dot{H}^1, \quad r^{-1}b_0 \in L^{3,2} \cap \dot{H}^1.$$

Assume that (ω, b) *is a regular solution of the system* (1.7), *then there hold*

$$\|\nabla b\|_{L^2}^2 + \|\partial_z \nabla b\|_{L^2_t(L^2)}^2 \leq C\|\nabla b_0\|_{L^2}^2 \exp\{CA_0(t) + CA_2(t)e^{CA_0(t)}\}, \tag{3.42}$$

where

$$\begin{aligned} A_2(t) &\overset{def}{=} t\|\omega_0\|_{L^{3,1}} + t^{\frac{3}{2}} \|\nabla(r^{-1}b_0)\|_{L^2}\|\nabla b_0\|_{L^2} + t\|\omega_0\|_{L^{3,1}}^2 \\ &\quad + t^2 \|\nabla(r^{-1}b_0)\|_{L^2}^2\|\nabla b_0\|_{L^2}^2 + \|\omega_0\|_{L^{\frac{3}{2},1}}^2 + t\,\|r^{-1}b_0\|_{L^{3,2}}^2\|b_0\|_{L^{3,2}}^2. \end{aligned}$$

Proof. Acting the operator ∂_r to the second of (1.7) yields

$$\partial_t\partial_r b + (u\cdot\nabla)\partial_r b - \partial_z^2\partial_r b = \partial_r(\frac{u^r b}{r}) - \partial_r u^r \partial_r b - \partial_r u^z \partial_z b. \tag{3.43}$$

Multiply (3.43) by $\partial_r b$, then integrating by part gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_r b\|_{L^2}^2 + \|\partial_z\partial_r b\|_{L^2}^2 &= \int_{\mathbb{R}^3} \partial_r(\frac{u^r b}{r})\partial_r b\,dx - \int_{\mathbb{R}^3} \partial_r u^r (\partial_r b)^2\,dx - \int_{\mathbb{R}^3} \partial_r u^z \partial_z b \partial_r b\,dx \\ &\triangleq K_1 + K_2 + K_3. \end{aligned} \tag{3.44}$$

By using of $\partial_r u^r = -\frac{u^r}{r} - \partial_z u^z$, we have

$$\begin{aligned} K_1 &= \int_{\mathbb{R}^3} -\frac{u^r}{r}\frac{b}{r}\partial_r b\,dx + \int_{\mathbb{R}^3} \partial_r u^r \frac{b}{r}\partial_r b\,dx + \int_{\mathbb{R}^3} \frac{u^r}{r}(\partial_r b)^2\,dx \\ &= -2\int_{\mathbb{R}^3} \frac{u^r}{r}\frac{b}{r}\partial_r b\,dx - \int_{\mathbb{R}^3} \partial_z u^z \frac{b}{r}\partial_r b\,dx + \int_{\mathbb{R}^3} \frac{u^r}{r}(\partial_r b)^2\,dx \\ &= -2\int_{\mathbb{R}^3} \frac{u^r}{r}\frac{b}{r}\partial_r b\,dx + \int_{\mathbb{R}^3} u^z \partial_z \frac{b}{r}\partial_r b\,dx + \int_{\mathbb{R}^3} u^z \frac{b}{r}\partial_z\partial_r b\,dx + \int_{\mathbb{R}^3} \frac{u^r}{r}(\partial_r b)^2\,dx, \end{aligned}$$

which by Hölder inequality infer to that

$$\begin{aligned} |K_1| &\leq 2\|\frac{u^r}{r}\|_{L^\infty}\|\frac{b}{r}\|_{L^2}\|\partial_r b\|_{L^2} + \|\frac{u^r}{r}\|_{L^\infty}\|\partial_r b\|_{L^2}^2 \\ &\quad + \|u^z\|_{L^\infty}\|\partial_z \frac{b}{r}\|_{L^2}\|\partial_r b\|_{L^2} + \|\partial_z\partial_r b\|_{L^2}\|u^z\|_{L^\infty}\|\frac{b}{r}\|_{L^2}. \end{aligned}$$

Along the same line, the bound of K_2 yields

$$\begin{aligned} K_2 &= \int_{\mathbb{R}^3} \frac{u^r}{r}(\partial_r b)^2\,dx + \int_{\mathbb{R}^3} \partial_z u^z (\partial_r b)^2\,dx = \int_{\mathbb{R}^3} \frac{u^r}{r}(\partial_r b)^2\,dx + 2\int_{\mathbb{R}^3} u^z \partial_z\partial_r b \partial_r b\,dx \\ &\leq \|\frac{u^r}{r}\|_{L^\infty}\|\partial_r b\|_{L^2}^2 + \|u^z\|_{L^\infty}\|\partial_z\partial_r b\|_{L^2}\|\partial_r b\|_{L^2}. \end{aligned}$$

By the definition $\partial_r u^z = \omega - \partial_z u^r$, we get $K_3 = \int_{\mathbb{R}^3} \omega\partial_z b\partial_r b\,dx - \int_{\mathbb{R}^3} \partial_z u^r \partial_z b \partial_r b\,dx$, which follows

$$|K_3| \lesssim (\|\omega\|_{L^3} + \|\partial_z u^r\|_{L^3})\|\partial_z b\|_{L^6}\|\partial_r b\|_{L^2}.$$

Due to the fact $\|\partial_z u^r\|_{L^3} \leq C\|\partial_z\omega\|_{L^{\frac{3}{2},3}}$, we find

$$|K_3| \lesssim (\|\omega\|_{L^3} + \|\partial_z\omega\|_{L^{\frac{3}{2},3}})\|\partial_z\nabla b\|_{L^2}\|\partial_r b\|_{L^2}.$$

Inserting the estimates of K_1-K_3 into (3.44), we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_r b\|_{L^2}^2 + \|\partial_z\partial_r b\|_{L^2}^2 &\leq C(\|r^{-1}u^r\|_{L^\infty} + \|u^z\|_{L^\infty}^2)\|(\partial_r b,\, r^{-1}b)\|_{L^2}^2 \\ &\quad + C(\|u^z\|_{L^\infty} + \|\omega\|_{L^3} + \|\partial_z\omega\|_{L^{\frac{3}{2},3}})\|(\partial_r b,\, r^{-1}b)\|_{L^2}\|\partial_z(\partial_r b,\, \partial_z b,\, r^{-1}b)\|_{L^2}, \end{aligned} \tag{3.45}$$

which along with (3.1) gives rise to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\partial_r b,\, r^{-1}b)\|_{L^2}^2 + \|\partial_z(\partial_r b,\, r^{-1}b)\|_{L^2}^2 \le C(\|r^{-1}u^r\|_{L^\infty} + \|u^z\|_{L^\infty}^2)\|(\partial_r b,\, r^{-1}b)\|_{L^2}^2 \\ + C(\|u^z\|_{L^\infty} + \|\omega\|_{L^3} + \|\partial_z\omega\|_{L^{\frac{3}{2},3}})\|(\partial_r b,\, r^{-1}b)\|_{L^2}\|\partial_z(\partial_r b,\, \partial_z b,\, r^{-1}b)\|_{L^2}.\end{aligned} \tag{3.46}$$

We may repeat the above argument to get the estimate of $\|\partial_z b\|_{L^2}$. In fact, acting the operator ∂_z to the second of the system (1.7) yields

$$\partial_t\partial_z b + (u\cdot\nabla)\partial_z b - \partial_z^2\partial_z b = \partial_z u^r(r^{-1}b - \partial_r b) + r^{-1}u^r\partial_z b - \partial_z u^z\partial_z b. \tag{3.47}$$

Multiply (3.47) by $\partial_z\, b$, then integrating by part gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_z b\|_{L^2}^2 + \|\partial_z^2 b\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3}\partial_z u^r(r^{-1}b - \partial_r b)\partial_z b\,\mathrm{d}x + \int_{\mathbb{R}^3} r^{-1}u^r(\partial_z b)^2\,\mathrm{d}x - \int_{\mathbb{R}^3}\partial_z u^z(\partial_z b)^2\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^3} u^r\partial_z[(r^{-1}b - \partial_r b)\partial_z b]\,\mathrm{d}x + \int_{\mathbb{R}^3} r^{-1}u^r(\partial_z b)^2\,\mathrm{d}x + 2\int_{\mathbb{R}^3} u^z\partial_z b\partial_z^2 b\,\mathrm{d}x.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_z b\|_{L^2}^2 + \|\partial_z^2 b\|_{L^2}^2 \\ &\le \|u^r\|_{L^\infty}\Big(\|\partial_z(r^{-1}b - \partial_r b)\|_{L^2}\|\partial_z b\|_{L^2} + \|(r^{-1}b - \partial_r b)\|_{L^2}\|\partial_z^2 b\|_{L^2}\Big) \\ &\quad + \Big(\|r^{-1}u^r\|_{L^\infty} + 2\|u^z\|_{L^\infty}^2\Big)\|\partial_z b\|_{L^2}^2,\end{aligned}$$

which along with (3.46) implies

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}^2 + \|\partial_z(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}^2 \\ &\le C\Big(\|r^{-1}u^r\|_{L^\infty} + \|u^z\|_{L^\infty}^2\Big)\|(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}^2 \\ &\quad + C\Big(\|u\|_{L^\infty} + \|\omega\|_{L^3} + \|\partial_z\omega\|_{L^{\frac{3}{2},3}}\Big)\|(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}\|\partial_z(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}.\end{aligned}$$

Thanks to Young's inequality, we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}^2 + \|\partial_z(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}^2 \\ &\le C\Big(\|(r^{-1}u^r,\, u^z)\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|\omega\|_{L^3}^2 + \|\partial_z\omega\|_{L^{\frac{3}{2},3}}^2\Big)\|(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}^2,\end{aligned}$$

which follows that

$$\begin{aligned}&\|(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L^2}^2 + \|\partial_z(\partial_r b,\, r^{-1}b,\, \partial_z b)\|_{L_t^2(L^2)}^2 \\ &\le C\|(\partial_r b_0,\, r^{-1}b_0,\, \partial_z b_0)\|_{L^2}^2 \\ &\quad \times \exp\Big\{C\int_0^t(\|(r^{-1}u^r,\, u^z)\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|\omega\|_{L^3}^2)\,\mathrm{d}\tau + C\|\partial_z\omega\|_{L_t^2(L^{\frac{3}{2},3})}^2\Big\}.\end{aligned} \tag{3.48}$$

Thanks to (3.18), we know that

$$\begin{aligned}&\|u\|_{L^\infty} + \|\omega\|_{L^3} \le C\|\omega\|_{L^{3,1}} \le C\big(\|\omega_0\|_{L^{3,1}} + \sqrt{t}\,\|r^{-1}b_0^2\|_{L^{3,1}}\big)e^{CA_0(t)}, \\ &\|\omega(t)\|_{L^{\frac{3}{2},1}} + \|\partial_z\omega\|_{L_t^2(L^{\frac{3}{2},1})} \le C\big(\|\omega_0\|_{L^{\frac{3}{2},1}} + \sqrt{t}\,\|r^{-1}b_0^2\|_{L^{\frac{3}{2},1}}\big)e^{CA_0(t)}.\end{aligned} \tag{3.49}$$

Hence, inserting (3.49) into (3.48) yields

$$\begin{aligned}&\|\nabla b\|_{L^2}^2+\|\partial_z\nabla b\|^2_{L^2_t(L^2)}\le C\|\nabla b_0\|^2_{L^2}e^{CA_0(t)}\\&\times\exp\{C\big(t\|\omega_0\|_{L^{3,1}}+t^{\frac32}\,\|r^{-1}b_0^2\|_{L^{3,1}}\\&\qquad+t\|\omega_0\|^2_{L^{3,1}}+t^2\,\|r^{-1}b_0^2\|^2_{L^{3,1}}+\|\omega_0\|^2_{L^{\frac32,1}}+t\,\|r^{-1}b_0^2\|^2_{L^{\frac32,1}}\big)e^{CA_0(t)}\}\end{aligned}$$

which implies (3.42).

Hence, we complete the proof of Proposition 3.4. □

Proposition 3.5. *Let the initial data (ω_0,b_0) satisfy*

$$\omega_0\in L^{3,1},\ r^{-1}\omega_0\in L^{\frac32,1},\ \partial_r\omega_0\in L^{\frac32},\ b_0\in L^{3,2}\cap\dot H^1,\ r^{-1}b_0\in H^1.$$

Let (ω,b) a regular solution of the system (1.7). *Then*

$$\|\partial_r\omega(t)\|_{L^\infty_t(L^{\frac32})}+\|\partial_z\partial_r\omega\|_{L^2_t(L^{\frac32})}\le C(t,\omega_0,b_0). \tag{3.50}$$

Proof. Multiplying the equation verified by $\partial_r\omega$ by $|\partial_r\omega|^{\frac12}\mathrm{sign}\ (\partial_r\omega)$ and integrating in space, we obtain

$$\begin{aligned}&\frac23\frac{\mathrm d}{\mathrm dt}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\frac89\|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2}\le2\|\frac{u^r}{r}\|_{L^\infty}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\int_{\mathbb R^3}\partial_zu^z|\partial_r\omega|^{\frac32}\mathrm dx\\&\quad+\big(2\|\frac{u^r}{r}\|_{L^\infty}\|\frac{\omega}{r}\|_{L^{\frac32}}+\|\partial_zu^z\frac{\omega}{r}\|_{L^{\frac32}}+\|g_1\|_{L^{\frac32}}\big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}+\int_{\mathbb R^3}\partial_z\partial_r(\frac{b^2}{r})|\partial_r\omega|^{\frac12}\mathrm dx,\end{aligned}$$

where $g_1:=-\partial_zu^r\partial_z\omega+\omega\partial_z\omega$. Integrating by parts and using the Cauchy-Schwartz inequality, we have

$$\int_{\mathbb R^3}\partial_zu^z|\partial_r\omega|^{\frac32}\mathrm dx=-2\int_{\mathbb R^3}u^z|\partial_r\omega|^{\frac34}\partial_z|\partial_r\omega|^{\frac34}\mathrm dx\le2\|u^z\|_{L^\infty}\Big\|\partial_z|\partial_r\omega|^{\frac34}\Big\|_{L^2}\|\partial_r\omega\|^{\frac34}_{L^{\frac32}}$$

and

$$\begin{aligned}&\int_{\mathbb R^3}\partial_z\partial_r(\frac{b^2}{r})|\partial_r\omega|^{\frac12}\,dx=-\int_{\mathbb R^3}\partial_z(\frac{b^2}{r^2})|\partial_r\omega|^{\frac12}\,dx+2\int_{\mathbb R^3}\frac br\partial_z\partial_rb|\partial_r\omega|^{\frac12}\mathrm dx+2\int_{\mathbb R^3}\partial_z\frac br\partial_rb|\partial_r\omega|^{\frac12}\mathrm dx\\&\lesssim\Big(\|\partial_z(\frac{b^2}{r^2})\|_{L^{\frac32}}+\|\frac br\|_{L^6}\|\partial_z\partial_rb\|_{L^2}+\|\partial_z\frac br\|_{L^6}\|\partial_rb\|_{L^2}\Big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}\\&\lesssim\Big(\|\partial_z(\frac{b^2}{r^2})\|_{L^{\frac32}}+\|\nabla\frac br\|_{L^2}\|\partial_z\partial_rb\|_{L^2}+\|\partial_z\nabla\frac br\|_{L^2}\|\partial_rb\|_{L^2}\Big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}.\end{aligned}$$

As a consequence, we have

$$\begin{aligned}&\frac{d}{dt}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2}\\&\lesssim\Big(\|\frac{u^r}{r}\|_{L^\infty}+\|u^z\|^2_{L^\infty}\Big)\|\partial_r\omega\|^{\frac32}_{L^{\frac32}}+\Big(\|\partial_zu^z\frac\omega r\|_{L^{\frac32}}+\|\frac{u^r}{r}\|_{L^\infty}\|\frac\omega r\|_{L^{\frac32}}+\|g_1\|_{L^{\frac32}}\Big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}\\&\quad+\Big(\|\partial_z(\frac{b^2}{r^2})\|_{L^{\frac32}}+\|\nabla\frac br\|_{L^2}\|\partial_z\partial_rb\|_{L^2}+\|\partial_z\nabla\frac br\|_{L^2}\|\partial_rb\|_{L^2}\Big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}.\end{aligned}\tag{3.51}$$

By Hölder's inequality, Proposition 2.2 and interpolation inequality, we have

$$\begin{aligned}\|\partial_zu^z\frac\omega r\|_{L^{\frac32}}&\lesssim\|\frac\omega r\|_{L_h^{\frac32}(L_v^\infty)}\|\partial_zu^z\|_{L_h^\infty(L_v^{\frac32})}\lesssim\|\frac\omega r\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac\omega r\|^{\frac23}_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}\Big(\|\partial_z\partial_r\omega\|^{\frac13}_{L^{\frac32}}+\|\partial_z\frac\omega r\|^{\frac13}_{L^{\frac32}}\Big)\\&\lesssim\|\frac\omega r\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac\omega r\|_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}+\|\frac\omega r\|^{\frac13}_{L^{\frac32}}\|\partial_z\frac\omega r\|^{\frac23}_{L^{\frac32}}\|\partial_z\omega\|^{\frac23}_{L^{\frac32}}\|\partial_z\partial_r\omega\|^{\frac13}_{L^{\frac32}},\end{aligned}$$

and consequently by Lemma 2.2 and Hölder's inequality, we obtain

$$\begin{aligned}\|\partial_z u^z \frac{\omega}{r}\|_{L^{\frac32}} \|\partial_r\omega\|^{\frac12}_{L^{\frac32}} \lesssim & \|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}} \|\partial_z \frac{\omega}{r}\|_{L^{\frac32}} \|\partial_z\omega\|^{\frac23}_{L^{\frac32}} \|\partial_r\omega\|^{\frac12}_{L^{\frac32}} \\ & + \|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}} \|\partial_z \frac{\omega}{r}\|^{\frac23}_{L^{\frac32}} \|\partial_z\omega\|^{\frac23}_{L^{\frac32}} \|\partial_z|\partial_r\omega|^{\frac34}\|^{\frac13}_{L^2} \|\partial_r\omega\|^{\frac{7}{12}}_{L^{\frac32}},\end{aligned}$$

and then

$$\begin{aligned}\|\partial_z u^z \frac{\omega}{r}\|_{L^{\frac32}} \|\partial_r\omega\|^{\frac12}_{L^{\frac32}} \leq & \varepsilon \|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2} + C\|\frac{\omega}{r}\|^{\frac13}_{L^{\frac32}} \|\partial_z \frac{\omega}{r}\|_{L^{\frac32}} \|\partial_z\omega\|^{\frac23}_{L^{\frac32}} \|\partial_r\omega\|^{\frac12}_{L^{\frac32}} \\ & + C_\varepsilon \|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}} \|\partial_z \frac{\omega}{r}\|^{\frac45}_{L^{\frac32}} \|\partial_z\omega\|^{\frac45}_{L^{\frac32}} \|\partial_r\omega\|^{\frac{7}{10}}_{L^{\frac32}}.\end{aligned}$$

On the other hand, thanks to Proposition 2.2 again, we find

$$\|g_1\|_{L^{\frac32}} \lesssim \|\partial_z u^r\|_{L^6} \|\partial_z\omega\|_{L^2} + \||\omega|^{\frac12}\|_{L^6} \|\partial_z|\omega|^{\frac32}\|_{L^2} \lesssim \|\partial_z\omega\|^2_{L^2} + \|\omega\|^{\frac12}_{L^3} \|\partial_z|\omega|^{\frac32}\|_{L^2}.$$

Thus in view of (3.51), we obtain

$$\begin{aligned}\frac{d}{dt}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}} + \|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2} \lesssim & \Big(\|\frac{u^r}{r}\|_{L^\infty} + \|u^z\|^2_{L^\infty}\Big)\|\partial_r\omega\|^{\frac32}_{L^{\frac32}} + \|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}} \|\partial_z \frac{\omega}{r}\|^{\frac45}_{L^{\frac32}} \|\partial_z\omega\|^{\frac45}_{L^{\frac32}} \|\partial_r\omega\|^{\frac{7}{10}}_{L^{\frac32}} \\ & + \Big(\|\frac{u^r}{r}\|_{L^\infty} \|\frac{\omega}{r}\|_{L^{\frac32}} + \|\partial_z\omega\|^2_{L^2} + \|\omega\|^{\frac12}_{L^3} \|\partial_z|\omega|^{\frac32}\|_{L^2} \\ & + \|\partial_z(\frac{b^2}{r^2})\|_{L^{\frac32}} + \|\nabla\frac{b}{r}\|_{L^2} \|\partial_z\partial_r b\|_{L^2} + \|\partial_z\nabla\frac{b}{r}\|_{L^2}\|\partial_r b\|_{L^2}\Big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}},\end{aligned}$$

which along with Lemma 2.2 and Proposition 2.2 implies

$$\begin{aligned}\frac{d}{dt}\|\partial_r\omega\|^{\frac32}_{L^{\frac32}} + \|\partial_z|\partial_r\omega|^{\frac34}\|^2_{L^2} \lesssim & \Big(\|\partial_z\frac{\omega}{r}\|_{L^{\frac32,1}} + \|\omega\|^2_{L^{3,1}} + \|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}} \|\partial_z \frac{\omega}{r}\|^{\frac45}_{L^{\frac32}} \|\partial_z\omega\|^{\frac45}_{L^{\frac32}}\Big)\|\partial_r\omega\|^{\frac32}_{L^{\frac32}} \\ & + \Big(\|\partial_z\frac{\omega}{r}\|_{L^{\frac32,1}} \|\frac{\omega}{r}\|_{L^{\frac32}} + \|\frac{\omega}{r}\|^{\frac25}_{L^{\frac32}} \|\partial_z \frac{\omega}{r}\|^{\frac45}_{L^{\frac32}} \|\partial_z\omega\|^{\frac45}_{L^{\frac32}} + \|\partial_z\omega\|^2_{L^2} + \|\omega\|^{\frac12}_{L^3} \|\partial_z|\omega|^{\frac32}\|_{L^2} \\ & + \|\partial_z(\frac{b^2}{r^2})\|_{L^{\frac32}} + \|\nabla\frac{b}{r}\|_{L^2} \|\partial_z\partial_r b\|_{L^2} + \|\partial_z\nabla\frac{b}{r}\|_{L^2}\|\partial_r b\|_{L^2}\Big)\|\partial_r\omega\|^{\frac12}_{L^{\frac32}}.\end{aligned}$$

Then Gronwall's inequality implies that

$$\begin{aligned} & \|\partial_r\omega\|_{L^\infty_t(L^{\frac32})} + \|\partial_z\partial_r\omega\|_{L^2_t(L^{\frac32})} \\ & \leq C\exp\{C\big(\|\partial_z\frac{\omega}{r}\|_{L^1_t(L^{\frac32,1})} + \|\omega\|^2_{L^2_t(L^{3,1})} + \|\frac{\omega}{r}\|^{\frac25}_{L^2_t(L^{\frac32})} \|\partial_z \frac{\omega}{r}\|^{\frac45}_{L^2_t(L^{\frac32})} \|\partial_z\omega\|^{\frac45}_{L^2_t(L^{\frac32})}\big)\} \\ & \times \Big(\|\partial_r\omega_0\|_{L^{\frac32}} + \|\partial_z\frac{\omega}{r}\|_{L^1_t(L^{\frac32,1})} \|\frac{\omega}{r}\|_{L^\infty_t(L^{\frac32})} + \|\frac{\omega}{r}\|^{\frac25}_{L^2_t(L^{\frac32})} \|\partial_z \frac{\omega}{r}\|^{\frac45}_{L^2_t(L^{\frac32})} \|\partial_z\omega\|^{\frac45}_{L^2_t(L^{\frac32})} \\ & \quad + \|\partial_z\omega\|^2_{L^2_t(L^2)} + \|\omega\|^{\frac12}_{L^\infty_t(L^3)} \|\partial_z|\omega|^{\frac32}\|_{L^1_t(L^2)} + \|\partial_z(\frac{b^2}{r^2})\|_{L^1_t(L^{\frac32})} \\ & \quad + \|\nabla\frac{b}{r}\|_{L^2_t(L^2)} \|\partial_z\partial_r b\|_{L^2_t(L^2)} + \|\partial_z\nabla\frac{b}{r}\|_{L^2_t(L^2)}\|\partial_r b\|_{L^2_t(L^2)}\Big).\end{aligned} \tag{3.52}$$

Therefore, inserting (3.38), (3.39), and (3.42) into (3.52) implies (3.50). We then complete the proof of Proposition 3.5. □

4. Proof of Theorem 1.1

4.1. **Existence part of the proof.** First of all, we note that ω_0, $r^{-1}\omega_0 \in L^{\frac32,1}(\mathbb{R}^3)$ which implies that u_0, $r^{-1}u_0 \in L^{3,1}(\mathbb{R}^3)$. Let $u_0 \in L^{3,1}(\mathbb{R}^3)$ be an axisymmetrical vector field without swirl such that $r^{-1}u_0 \in L^{3,1}(\mathbb{R}^3)$, $\omega_0 \in L^{\frac32,1}(\mathbb{R}^3)$, and $r^{-1}\omega_0 \in L^{\frac32,1}(\mathbb{R}^3)$, and assume that the initial axisymmetric data $b_0 \in L^{3,2}(\mathbb{R}^3)$ with $r^{-1}b_0 \in L^{3,2}(\mathbb{R}^3)$.

Let J_n the operator which localizes in low frequencies defined by

$$\widehat{J_n f}(\xi) \stackrel{\text{def}}{=} \chi(2^{-n}\xi)\widehat{f}(\xi) \quad (\forall\, n \in \mathbb{Z}),$$

where $\chi(\xi)$ is a radial and regular function, equal to which to 1 on a ball around zero, and $\widehat{f}(\xi)$ is the Fourier transform of f. Since $(u_0,\, B_0)$ is axisymmetrical with the form (1.4), we know that $(J_n u_0,\, J_n B_0)$ is also axisymmetrical with the form (1.4) and also is regular (see for example [2]). So, by [30], there exists a unique regular and global in time axisymmetrical solution $(u^n,\, B^n)$ (with the form (1.4)) to the system

$$\begin{cases} \partial_t u^n + u^n\cdot\nabla u^n - \nu_z\partial_z^2 u^n + \nabla\Pi^n = B^n\cdot\nabla B^n,\\ \partial_t b^n + u^n\cdot\nabla b^n - \mu_z\partial_z^2 B^n = B^n\cdot\nabla u^n,\\ \operatorname{div} u^n = \operatorname{div} B^n = 0,\\ (u^n,\, b^n)|_{t=0} = (J_n u_0,\, J_n B_0), \end{cases}$$

that is,

$$\begin{cases} \partial_t\omega^n + \nabla\cdot(\omega^n u^n) - \frac{(u^n)^r}{r}\omega^n - \partial_z^2\omega^n = -\partial_z\big(\frac{(b^n)^2}{r}\big),\\ \partial_t\frac{b^n}{r} + \nabla\cdot\big(\frac{b^n}{r}u^n\big) - \partial_z^2\frac{b^n}{r} = 0,\\ u^n = (-\Delta)^{-1}\nabla\times\,(\omega^n e_\theta),\\ (\omega^n,\, b^n)|_{t=0} = (J_n\omega_0,\, J_n b_0). \end{cases} \tag{4.1}$$

Notice that $J_n\omega_0$ and $\frac{J_n\omega_0}{r}$ are uniformly bounded in $L^{\frac{3}{2},1}(\mathbb{R}^3)$, and $J_n b_0$ and $\frac{J_n b_0}{r}$ are uniformly bounded in $L^{3,2}(\mathbb{R}^3)$, we then obtain from Propositions 3.1 and 3.2 that:

$$\begin{aligned} &\{(u^n,\omega^n,b^n)\}_{n\in\mathbb{N}} \quad \text{is uniformly bounded (u.b. for short) in}\\ &\qquad\qquad L^\infty_{loc}(\mathbb{R}^+;\dot W^{1,\frac{3}{2}})\times L^\infty_{loc}(\mathbb{R}^+;L^{\frac{3}{2},1})\times L^\infty_{loc}(\mathbb{R}^+;L^{3,2});\\ &\{(\frac{(u^n)^r}{r},\frac{b^n}{r})\}_{n\in\mathbb{N}} \quad \text{is u.b. in} \quad L^\infty_{loc}(\mathbb{R}^+;L^{3,1})\times L^\infty_{loc}(\mathbb{R}^+;L^{3,2});\\ &\{(\partial_z u^n,\partial_z\omega^n)\}_{n\in\mathbb{N}} \quad \text{is u.b. in} \quad L^2_{loc}(\mathbb{R}^+;W^{1,\frac{3}{2}})\times L^2_{loc}(\mathbb{R}^+;L^{\frac{3}{2},1});\\ &\{\frac{(u^n)^r}{r}\}_{n\in\mathbb{N}} \quad \text{is u.b. in} \quad L^2_{loc}(\mathbb{R}^+;\,L^\infty);\\ &\{(\partial_t u^n,\partial_t\frac{b^n}{r})\}_{n\in\mathbb{N}} \quad \text{is u.b. in} \quad L^1_{loc}(\mathbb{R}^+;L^{\frac{3}{2}})\times L^1_{loc}(\mathbb{R}^+;\dot W^{-1,\frac{3}{2}}). \end{aligned} \tag{4.2}$$

By standard compactness arguments and the Arzela-Ascoli lemma, we can obtain up to a subsequence denoted again by (u^n,b^n), that (u^n,b^n) converges strongly to (u,b) in $C_{loc}(\mathbb{R}^+;L^2_{loc})\times C_{loc}(\mathbb{R}^+;\dot H^{-\frac{1}{2}}_{loc})$. Interpolating with the fact that $(u_n,\,\frac{b^n}{r})$ has uniform bound in (4.2), we found that $u_n\to u$ in $L^2_{loc}(\mathbb{R}^+;H^{\frac{1}{4}}_{loc}(\mathbb{R}^3))$ and $\frac{b^n}{r}\to\frac{b}{r}$ in $L^2_{loc}(\mathbb{R}^+;L^{\frac{3}{2}}_{loc}(\mathbb{R}^3))$. This allows to pass to the limit in the nonlinear terms and we conclude that $(\omega^n u^n\to\omega u,\ \frac{b^n}{r}u^n\to\frac{b}{r}u,\ \frac{b^n}{r}b^n\to\frac{b}{r}b,\ \frac{(u^n)^r b^n}{r}\to\frac{u^r b}{r})$ in $\mathcal{D}'$. Finally, by passing to the limit in the system (4.1) we obtain a global in time, axisymmetric solution, without swirl, (u,B) of the system (1.5).

4.2. **Uniqueness part of the proof.** In order to prove the uniqueness of the solution for the system (1.7), let $(\omega_1,\,b_1)$ and $(\omega_2,\,b_2)$ be two solutions, and define $(\delta\omega,\delta b)\stackrel{\text{def}}{=}(\omega_2-\omega_1,b_2-b_1)$ their differences, which verifies the following system:

$$\begin{cases} \partial_t\delta\omega + (u_2\cdot\nabla)\delta\omega - \partial_z^2\delta\omega = -(\delta u\cdot\nabla)\omega_1 + \frac{u_2^r}{r}\delta\omega + \frac{\delta u^r}{r}\omega_1 + \partial_z[\delta b(\frac{b_1+b_2}{r})],\\ \partial_t\delta b + (u_2\cdot\nabla)\delta b - \partial_z^2\delta b = -(\delta u\cdot\nabla)b_1 + \frac{u_2^r}{r}\delta b + \frac{\delta u^r}{r}b_1,\\ (\delta\omega,\delta b)_{|t=0} = (0,0). \end{cases} \tag{4.3}$$

The functional framework where we control the differences of the two solutions is L^p with $\frac{6}{5} \le p < \frac{3}{2}$. The energy estimates imply that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\omega\|_{L^p}^p+\frac{4(p-1)}{p^2}\Big\|\partial_z|\delta\omega|^{\frac{p}{2}}\Big\|_{L^2}^2 &\le \|\frac{u_2^r}{r}\|_{L^\infty}\|\delta\omega\|_{L^p}^p+\|\frac{\omega_1\delta u^r}{r}\|_{L^p}\|\delta\omega\|_{L^p}^{p-1}\\&+\|(\delta u\cdot\nabla)\omega_1\|_{L^p}\|\delta\omega\|_{L^p}^{p-1}+\|\partial_z\delta b\|_{L^{\frac{6p}{6-p}}}\|\frac{b_1+b_2}{r}\|_{L^6}\|\delta\omega\|_{L^p}^{p-1}\\&+\|\delta b\|_{L^{\frac{6p}{6-p}}}\|\partial_z\frac{b_1+b_2}{r}\|_{L^6}\|\delta\omega\|_{L^p}^{p-1}.\end{aligned}$$

Using Hölder inequality, Sobolev embedding, Proposition 2.2 and Lemma 2.1, we have

$$\begin{aligned}&\|\frac{\omega_1\delta u^r}{r}\|_{L^p}+\|(\delta u\cdot\nabla)\omega_1\|_{L^p}\\&\le\big(\|\frac{\omega_1}{r}\|_{L^{\frac{3}{2}}}+\|\partial_r\omega_1\|_{L^{\frac{3}{2}}}\big)\|\delta u^r\|_{L^{\frac{3p}{3-2p}}}+\|\partial_z\omega_1\|_{L^6_h(L^{\frac{3}{2}}_v)}\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}\\&\lesssim\big(\|\frac{\omega_1}{r}\|_{L^{\frac{3}{2}}}+\|\partial_r\omega_1\|_{L^{\frac{3}{2}}}\big)\|\partial_z\delta\omega\|_{L^p}+\|\partial_z\partial_r\omega_1\|_{L^{\frac{3}{2}}}\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}\\&\lesssim\Big(\|\frac{\omega_1}{r}\|_{L^{\frac{3}{2}}}+\|\partial_r\omega_1\|_{L^{\frac{3}{2}}}\Big)\Big\|\partial_z|\delta\omega|^{\frac{p}{2}}\Big\|_{L^2}\|\delta\omega\|_{L^p}^{\frac{2-p}{2}}+\|\partial_z\partial_r\omega_1\|_{L^{\frac{3}{2}}}\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}.\end{aligned}$$

Concerning $\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}$ and using the identity $\Delta\delta u^z=\partial_r\delta\omega+r^{-1}\delta\omega$, we obtain by integration by parts that $|\delta u^z|\lesssim\frac{1}{|\cdot|^2}\star|\delta\omega|$. Then, using the convolution law, we obtain

$$\|\delta u^z\|_{L_h^{\frac{6p}{6-p}}(L_v^{\frac{3p}{3-2p}})}\lesssim\|\delta\omega\|_{L_h^{p,\frac{6p}{6-p}}(L_v^p)}\lesssim\|\delta\omega\|_{L^p}.$$

The Young inequality implies that

$$\begin{aligned}\frac{d}{dt}\|\delta\omega\|_{L^p}^p+\frac{2(p-1)}{p}\Big\|\partial_z|\delta\omega|^{\frac{p}{2}}\Big\|_{L^2}^2&\lesssim\Big(\|\frac{u_2^r}{r}\|_{L^\infty}+\|\frac{\omega_1}{r}\|_{L^{\frac{3}{2}}}^2+\|\partial_r\omega_1\|_{L^{\frac{3}{2}}}^2+\|\partial_z\partial_r\omega_1\|_{L^{\frac{3}{2}}}\Big)\|\delta\omega\|_{L^p}^p\\&+\Big(\|\partial_z\delta b\|_{L^{\frac{6p}{6-p}}}\|(\frac{b_1}{r},\,\frac{b_2}{r})\|_{L^6}+\|\delta b\|_{L^{\frac{6p}{6-p}}}\|\partial_z(\nabla\frac{b_1}{r},\,\nabla\frac{b_2}{r})\|_{L^2}\Big)\|\delta\omega\|_{L^p}^{p-1},\end{aligned}$$

which follows that

$$\begin{aligned}&\|\delta\omega\|_{L_t^\infty(L^p)}^p+\Big\|\partial_z|\delta\omega|^{\frac{p}{2}}\Big\|_{L_t^2(L^2)}^2\le C\|f_1(\tau)\|_{L^1([0,t])}\|\delta\omega\|_{L_t^\infty(L^p)}^p\\&+C\Big(\|\partial_z\delta b\|_{L_t^2(L^{\frac{6p}{6-p}})}\|(\frac{b_1}{r},\,\frac{b_2}{r})\|_{L_t^2(L^6)}+\|\delta b\|_{L_t^\infty(L^{\frac{6p}{6-p}})}\|\partial_z(\nabla\frac{b_1}{r},\,\nabla\frac{b_2}{r})\|_{L_t^1(L^2)}\Big)\|\delta\omega\|_{L_t^\infty(L^p)}^{p-1},\end{aligned}$$

where

$$\mathcal{F}_1(t)\overset{\text{def}}{=}\|\frac{u_2^r}{r}\|_{L_t^1(L^\infty)}+\|\frac{\omega_1}{r}\|_{L_t^2(L^{\frac{3}{2}})}^2+\|\partial_r\omega_1\|_{L_t^2(L^{\frac{3}{2}})}^2+\|\partial_z\partial_r\omega_1\|_{L_t^1(L^{\frac{3}{2}})}. \tag{4.4}$$

Hence, due to $\mathcal{F}_1(t)\to0$ as $t\to0^+$, we get that, there is $\epsilon_1>0$ so small that, if $t\in[0,\epsilon_1]$, there hold

$$\|\delta\omega\|_{L_t^\infty(L^p)}+\Big\|\partial_z|\delta\omega|^{\frac{p}{2}}\Big\|_{L_t^2(L^2)}^{\frac{2}{p}}\le C\mathcal{F}_2(t)\|\partial_z\delta b\|_{L_t^2(L^{\frac{6p}{6-p}})}+C\mathcal{F}_3(t)\|\delta b\|_{L_t^\infty(L^{\frac{6p}{6-p}})}, \tag{4.5}$$

where

$$\mathcal{F}_2(t)\overset{\text{def}}{=}\|(\frac{b_1}{r},\,\frac{b_2}{r})\|_{L_t^2(L^6)},\quad\mathcal{F}_3(t)\overset{\text{def}}{=}\|\partial_z(\nabla\frac{b_1}{r},\,\nabla\frac{b_2}{r})\|_{L_t^1(L^2)}. \tag{4.6}$$

Due to (2.1), we arrive at

$$\begin{aligned}&\|\delta\omega\|_{L^\infty_t(L^p)}+\|\partial_z\delta\omega\|_{L^2_t(L^p)}\\&\le C\mathcal{F}_2(t)\|\partial_z|\delta b|^{\frac{3p}{6-p}}\|_{L^2_t(L^2)}\|\delta b\|^{1-\frac{3p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}+C\mathcal{F}_3(t)\|\delta b\|_{L^\infty_t(L^{\frac{6p}{6-p}})}.\end{aligned}\tag{4.7}$$

On the other hand, from the δb equation, the energy estimates imply

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\delta b\|^{\frac{6p}{6-p}}_{L^{\frac{6p}{6-p}}}+\Big\|\partial_z|\delta b|^{\frac{3p}{6-p}}\Big\|^2_{L^2}\\&\lesssim\big(\|(\delta u\cdot\nabla)b_1\|_{L^{\frac{6p}{6-p}}}+\|\frac{u_2^r}{r}\|_{L^\infty}\|\delta b\|_{L^{\frac{6p}{6-p}}}+\|\frac{\delta u^r}{r}b_1\|_{L^{\frac{6p}{6-p}}}\big)\|\delta b\|^{\frac{7p-6}{6-p}}_{L^{\frac{6p}{6-p}}}.\end{aligned}\tag{4.8}$$

Again, using Hölder inequality, Proposition 2.2, Sobolev embedding, we have

$$\begin{aligned}&\|(\delta u\cdot\nabla)b_1\|_{L^{\frac{6p}{6-p}}}\lesssim\|\partial_z\delta\omega\|_{L^p}\|\partial_r b_1\|_{L^2}+\|\delta\omega\|_{L^p}\|\partial_z\nabla b_1\|_{L^2},\\&\|\frac{\delta u^r}{r}b_1\|_{L^{\frac{6p}{6-p}}}\lesssim\|\delta u^r\|_{L^{\frac{3p}{3-2p}}}\|\frac{b_1}{r}\|_{L^2}\lesssim\|\partial_z\delta\omega\|_{L^p}\|\frac{b_1}{r}\|_{L^2}.\end{aligned}$$

Notice that

$$\begin{aligned}\|(\delta u\cdot\nabla)b_1\|_{L^{\frac{6p}{6-p}}}&\lesssim\|\delta u^r\partial_r b_1\|_{L^{\frac{6p}{6-p}}}+\|\delta u^z\partial_z b_1\|_{L^{\frac{6p}{6-p}}}\\&\lesssim\|\delta u^r\|_{L^{\frac{3p}{3-2p}}}\|\partial_r b_1\|_{L^2}+\|\delta u^z\|_{L^{\frac{3p}{3-p}}}\|\partial_z b_1\|_{L^6}.\end{aligned}\tag{4.9}$$

Form Proposition 2.2, we known that, for $\frac{6}{5}\le p<\frac{3}{2}$,

$$\|\delta u^r\|_{L^{\frac{3p}{3-2p}}}\lesssim\|\partial_z\delta\omega\|_{L^p},\quad\|\delta u^z\|_{L^{\frac{3p}{3-p}}}\lesssim\|\delta\omega\|_{L^p},$$

which along with (4.9) implies

$$\|(\delta u\cdot\nabla)b_1\|_{L^{\frac{6p}{6-p}}}\lesssim\|\partial_z\delta\omega\|_{L^p}\|\partial_r b_1\|_{L^2}+\|\delta\omega\|_{L^p}\|\partial_z\nabla b_1\|_{L^2}.$$

For the last term of (4.8), again using the above embedding inequality, we obtain

$$\|\frac{\delta u^r}{r}b_1\|_{L^{\frac{6p}{6-p}}}\lesssim\|\delta u^r\|_{L^{\frac{3p}{3-2p}}}\|\frac{b_1}{r}\|_{L^2}\lesssim\|\partial_z\delta\omega\|_{L^p}\|\frac{b_1}{r}\|_{L^2}$$

and then, we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta b\|^{\frac{6p}{6-p}}_{L^{\frac{6p}{6-p}}}+\Big\|\partial_z|\delta b|^{\frac{3p}{6-p}}\Big\|^2_{L^2}&\lesssim\big(\|\partial_z\delta\omega\|_{L^p}\|\partial_r b_1\|_{L^2}+\|\delta\omega\|_{L^p}\|\partial_z\nabla b_1\|_{L^2}\\&\quad+\|\frac{u_2^r}{r}\|_{L^\infty}\|\delta b\|_{L^{\frac{6p}{6-p}}}+\|\partial_z\delta\omega\|_{L^p}\|\frac{b_1}{r}\|_{L^2}\big)\|\delta b\|^{\frac{7p-6}{6-p}}_{L^{\frac{6p}{6-p}}}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\|\delta b\|^{\frac{6p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}+\Big\|\partial_z|\delta b|^{\frac{3p}{6-p}}\Big\|^2_{L^2_t(L^2)}\le C\|r^{-1}u_2^r\|_{L^1_t(L^\infty)}\|\delta b\|^{\frac{6p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}\\&+C(\|(\partial_r b_1,\frac{b_1}{r})\|_{L^2_t(L^2)}+\|\partial_z\nabla b_1\|_{L^1_t(L^2)})(\|\partial_z\delta\omega\|_{L^2_t(L^p)}+\|\delta\omega\|_{L^\infty_t(L^p)})\|\delta b\|^{\frac{7p-6}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}.\end{aligned}\tag{4.10}$$

Substituting (4.7) into (4.10) gives rise to

$$\begin{aligned}&\|\delta b\|^{\frac{6p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}+\Big\|\partial_z|\delta b|^{\frac{3p}{6-p}}\Big\|^2_{L^2_t(L^2)}\le C\|r^{-1}u_2^r\|_{L^1_t(L^\infty)}\|\delta b\|^{\frac{6p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}\\&+C\mathcal{F}_4(t)\bigg(\mathcal{F}_2(t)\|\partial_z|\delta b|^{\frac{3p}{6-p}}\|_{L^2_t(L^2)}\|\delta b\|^{1-\frac{3p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}+\mathcal{F}_3(t)\|\delta b\|_{L^\infty_t(L^{\frac{6p}{6-p}})}\bigg)\|\delta b\|^{\frac{7p-6}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}\end{aligned}$$

with

$$\mathcal{F}_4(t) \stackrel{\text{def}}{=} \|(\partial_r b_1, \frac{b_1}{r})\|_{L^2_t(L^2)} + \|\partial_z \nabla\, b_1\|_{L^1_t(L^2)}, \tag{4.11}$$

which along with Young's inequality implies

$$\|\delta b\|^{\frac{6p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})} + \left\|\partial_z |\delta b|^{\frac{3p}{6-p}}\right\|^2_{L^2_t(L^2)} \leq C\mathcal{F}_5(t)\|\delta b\|^{\frac{6p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})}$$

with

$$\mathcal{F}_5(t) \stackrel{\text{def}}{=} \|r^{-1}u_2^r\|_{L^1_t(L^\infty)} + \mathcal{F}_3(t)\mathcal{F}_4(t) + (\mathcal{F}_2(t)\mathcal{F}_4(t))^2. \tag{4.12}$$

Notice that $\mathcal{F}_5(t) \to 0$ as $t \to 0+$, so we get that, there exists $\epsilon_0 \in (0, \epsilon_1)$ so small that, if $t \in [0, \epsilon_0]$, there holds

$$\|\delta b\|^{\frac{6p}{6-p}}_{L^\infty_t(L^{\frac{6p}{6-p}})} + \left\|\partial_z |\delta b|^{\frac{3p}{6-p}}\right\|^2_{L^2_t(L^2)} = 0,$$

which immediately follows from (4.5) and (2.1) that

$$\|\delta\omega\|_{L^\infty_t(L^p)} = 0.$$

Therefore, we obtain $\delta\, b(t) = \delta\omega(t) \equiv 0$ for any $t \in [0, \epsilon_0]$. The uniqueness of such strong solutions on the whole time interval $[0, +\infty)$ then follows by a bootstrap argument.

Moreover, the continuity with respect to the initial data may also be obtained by the same argument in the proof of the uniqueness, which ends the proof of Theorem 1.1.

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