GLOBAL WELL-POSEDNESS FOR THE 2-D INHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES SYSTEM WITH LARGE INITIAL DATA IN CRITICAL SPACES

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Abstract. Without any smallness assumption, we prove the global unique solvability of the 2-D incompressible inhomogeneous Navier-Stokes equations with initial data in the critical Besov space, which is almost the energy space in the sense that they have the same scaling in terms of this 2-D system.

Keywords: Inhomogeneous Navier-Stokes equations, Global well-posedness, Critical spaces

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1. Introduction

We consider in this paper the Cauchy problem of the following 2-D incompressible inhomogeneous Navier-Stokes equations

$$
\begin{cases}
\partial_t\rho + \mathrm{div}(\rho u) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2,\\
\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\mu\, \mathbb{D}(u)) + \nabla\Pi = 0,\\
\mathrm{div}\, u = 0,\\
\rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = m_0,
\end{cases}
\tag{1.1}
$$

where ρ and $u = (u_1, u_2)^T$ stand for the density and velocity of the fluid respectively, $\mathbb{D}(u) = \nabla u + \nabla u^T$, Π is a scalar pressure function, and in general, the viscous coefficient $\mu = \mu(\rho)$ is a smooth, positive function on $[0, \infty)$. Such system describes a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [21] for the detailed derivation.

If $\mu(\rho)$ is independent of ρ, that is, μ is a positive constant (taking $\mu = 1$ for simplicity), then the system is rewritten as the form

$$
\begin{cases}
\partial_t\rho + \mathrm{div}(\rho u) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2,\\
\rho(\partial_t u + (u\cdot\nabla)u) - \Delta u + \nabla\Pi = 0,\\
\mathrm{div}\, u = 0,\\
(\rho, u)|_{t=0} = (\rho_0, u_0).
\end{cases}
\tag{1.2}
$$

When ρ_0 is bounded away from 0, Kazhikov [5] proved the global existence of strong solutions to the system (1.2) for smooth initial data in two dimensions, also proved that the system (1.1) has at least one global weak solutions in the energy space. However, the uniqueness of this type weak solutions has not be solved. Considering the case of the bounded domain Ω with homogeneous Dirichlet boundary condition for the fluid velocity, Ladyvzenskaja and Solonnikov [20] first addressed the question of unique solvability of (1.1). In particular, under the assumptions that $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$ $(p > 2)$ is divergence free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\Omega)$ is bounded away from zero, then they [20] proved global well-posedness of (1.1). Similar results were obtained by Danchin [13] in $\mathbb{R}^2$ with initial data in almost critical Sobolev spaces.

In general, DiPerna and Lions [17, 21] proved the global existence of weak solutions to (1.1) in energy space in any space dimensions. Yet the uniqueness and regularities of such weak solutions are big open questions even in two space dimension, as was mentioned by Lions in [21].

On the other hand, if the density ρ is away from zero, we denote by $a \stackrel{\text{def}}{=} \rho^{-1} - 1$, then the system (1.2) can be equivalently reformulated as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \Delta\, u) = 0, \\ \operatorname{div} u = 0, \\ (a,u)|_{t=0} = (a_0, u_0). \end{cases} \tag{1.3}$$

Just as the classical Navier-Stokes system, which is the case when $a = 0$ in (1.3), the system (1.3) also has a scaling-invariant transformation. Indeed if (a,u) solves (1.3) with initial data (a_0, u_0), then for $\forall\, \ell > 0$,

$$(a,u)_\ell(t,x) \stackrel{\text{def}}{=} (a(\ell^2 \cdot, \ell \cdot), \ell u(\ell^2 \cdot, \ell \cdot)) \tag{1.4}$$

is also a solution of (1.3) with initial data $(a_0(\ell \cdot), \ell u_0(\ell \cdot))$. Some results about global existence and uniqueness of the solutions in critical spaces for small data were proved in [1, 3, 16]. Recently, we [2] first investigated the well-posedness of the 3-D incompressible inhomogeneous Navier-Stokes equation (1.2) with initial data (a_0, u_0) in the critical spaces and without size restriction on a_0.

For the two-dimensional case, when the density and the velocity have more regularity, Danchin [13] proved the global well-poedness result of the system (1.2). More precisely, if $0 < m \leq \rho_0 \leq M$, $\rho_0^{-1} - 1 \in H^{1+\alpha}$ and $u_0 \in H^\beta$ with $\alpha, \beta > 0$, the system (1.2) is globally well-posed. Recently, some improvements of this result have been achieved. Paicu, Zhang, and Zhang [22] investigated the unique solvability of the global solution of the 2-D system (1.2) if $0 < m \leq \rho_0 \leq M$ and $u_0 \in H^s$ with $s > 0$, and the first author in the paper and Zhang [4] proved the global existence and uniqueness of the solution to the 2-D system (1.2) if $0 < m \leq \rho_0 \leq M$, $\rho_0^{-1} - 1 \in \dot{B}^1_{2,1} \cap \dot{B}^\alpha_{\infty,\infty}$ with $\alpha > 0$ and $u_0 \in \dot{B}^0_{2,1}$, which is some extension of the results in the 2-D homogeneous incompressible Navier-Stokes equations (see [6] for instant). In fact, it is well known that there is the gain of two derivatives in $L^1_{loc}(\mathbb{R}^+, \dot{B}^2_{2,1})$ starting from $\dot{B}^0_{2,1}$ initial velocity in the the 2-D homogeneous incompressible Navier-Stokes equations ([6]). Without a positive lower bound of the density, Danchin and Mucha [15] studied the existence and uniqueness of global solution to (1.2) if $0 \leq \rho_0 \leq M$, $\int_{\mathbb{R}^2} \rho_0 > 0$ and $u_0 \in H^1$. It's worth to mention that Haspot [19] proved the global well-posedness of the system (1.2) with small non-Lipschitz velocity $u_0 \in \dot{B}^{\frac{2}{p_2}-1}_{p_2,r}$ and more regular density $\rho_0^{-1} - 1 \in B^{\frac{2}{p_1}+\varepsilon}_{p_1,\infty}$ under some restrictions on p_1, p_2, r, and ε.

In summary, all the well-posedness results of the 2-D system (1.2) obtained so far are under the additional assumption that the density or the velocity has more regularity compared to critical spaces.

In this paper, we investigate the global well-posedness of the 2-D inhomogeneous incompressible Navier-Stokes system (1.2) with large initial data in the critical space, which is **almost** the energy space in the sense that they have the same scaling in terms of the system (1.1) (see Remark 1.1 below).

The main theorem of the paper is stated as follows.

Theorem 1.1. *Assume that m, M are two positive constants and ε is a positive constant in $(0,1)$. Let $u_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ be a solenoidal vector field and $\rho_0^{-1} - 1 \in \dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}(\mathbb{R}^2)$ satisfy*

$$m \leq \rho_0 \leq M. \tag{1.5}$$

Then the 2-D system (1.2) *has a global solution* $(\rho, u, \nabla\Pi)$ *with*

$$\begin{aligned}
&\rho^{-1} - 1 \in C(\mathbb{R}_+;\, \dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}(\mathbb{R}^2)) \\
&u \in C(\mathbb{R}_+;\, \dot{B}^{0}_{2,1}(\mathbb{R}^2)) \cap L^1_{loc}(\mathbb{R}_+;\, \dot{B}^{2}_{2,1}(\mathbb{R}^2)) \quad \text{and} \\
&\partial_t u,\, \nabla\Pi \in L^1_{loc}(\mathbb{R}_+;\, \dot{B}^{0}_{2,1}(\mathbb{R}^2)).
\end{aligned} \tag{1.6}$$

Furthermore, if, in addition, $\rho_0^{-1} - 1 \in \dot{B}^1_{2,1}$, *then the 2-D system* (1.2) *is globally well-posed.*

Remark 1.1. *Compared with the global existence theorem of weak solutions to the system* (1.2) *in the energy space (*[5, 17, 21]*) (where initial data satisfies* $\rho_0 - 1 \in L^\infty(\mathbb{R}^2)$ *and* $u_0 \in L^2(\mathbb{R}^2)$*), Theorem 1.1 requires that the initial density* $\rho_0 - 1 \in \dot{B}^1_{2,1}(\mathbb{R}^2))$ *which has the same scaling as* $\rho_0 - 1 \in L^\infty(\mathbb{R}^2)$ *in the sense of the scaling-invariant transformation* (1.4) *of the system* (1.2)*, and the initial velocity* $u_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ *which has the same scaling and regularity as* $u_0 \in L^2(\mathbb{R}^2)$ *in the energy space.*

The proof of Theorem 1.1 is completed in Sections 2-4. We now present a summary of principal difficulties we encounter in our analysis as well as a sketch of the key ideas used in our proof.

The first difficulty to the proof of Theorem 1.1 lies in the fact that when a is not small, we can not use the classical arguments in [1, 3] to deal with the following linearized momentum equations of (1.3):

$$\partial_t u - (1 + a)(\Delta u - \nabla\Pi) = f, \tag{1.7}$$

Motivated by [11] and [2], for some large enough integer m, we shall rewrite (1.7) as

$$\partial_t u - (1 + \dot{S}_m a)(\Delta u - \nabla\Pi) = (a - \dot{S}_m a)(\Delta u - \nabla\Pi) + f, \tag{1.8}$$

with the low frequency part $\dot{S}_m a$ of a (defined in (2.2)). Then the basic energy method can be used to solve (1.8) when we deal with the global existence of the solution to (1.3).

The other difficulty in the proof of Theorem 1.1 is how to deal with the uniqueness issue of the solution. In order to solve this problem, the crucial part is, roughly speaking, to control the *Lip* norm of the velocity u, which will conserve all the regularities of the density and the velocity in critical spaces, as well as the smallness of the high frequency part $a - \dot{S}_m a$ of a with m being large enough.

As we mentioned above, if the density or the velocity has more regularity than the data in the critical space, combining the losing estimates for transport equations with the theory of transport-diffusion equations [6] provides the boundness of the *Lip* norm of the velocity, which will in turn close the estimates in the proof of global well-posedness of (1.2) (see [22, 4, 15]).

In our critical case, there is no more regularity of the density or the velocity to rescue their losing regularity when we solve the transport equation of the density or the transport-diffusion equations in terms of the velocity.

For this reason, we need first to get, at least in a small time interval, the $L^1([0,T]; \dot{B}^2_{2,1})$ estimate for the velocity field (see Proposition 3.2), which relies on more elaborate application of Littlewood-Paley theory, as well as the basic energy and the estimate of $\|\nabla\Pi\|_{L^1_T(L^2)}$. Based on this, together with Osgood's lemma applied, we solve the uniqueness issue of the solution to (1.2) in the critical space. The global well-posedness of the system (1.2) then will be obtained from the result in [22] as well as the smoothness of the velocity in the dissipative system (1.2).

The rest of the paper is organized as follows. In Section 2, we recall some basic ingredients of Littlwood-Paley theory, and derive some qualitative and analytic properties of the flow, as well as some necessary commutator estimates. We then prove the $L^1([0,T]; \dot{B}^2_{2,1})$ estimate for the velocity field in Section 3. Finally, the proof of Theorem 1.1 is completed in Section 4.

Notations: Let A, B be two operators, we denote $[A,B]=AB-BA$, the commutator between A and B. By $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leq Cb$ and C_0 denotes a positive constant depending only on the initial data. And $a\sim b$ means that both $a\lesssim b$ and $b\lesssim a$. For $r\in[1,+\infty]$, we denote $\{c_{j,r}\}_{j\in\mathbb{Z}}$ (or $\{c_{j,r}\}_{j\in\mathbb{N}\cup\{-1\}}$) a sequence in $\ell^r(\mathbb{Z})$ (or $\ell^r(\mathbb{N}\cup\{-1\})$) such that $\|\{c_{j,r}\}_j\|_{\ell^r}=1$.

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;\,X)$ the set of continuous functions on I with values in X, and by $\mathcal{C}_b(I;\,X)$ the subset of bounded functions of $\mathcal{C}(I;\,X)$. For $q\in[1,+\infty]$, the notation $L^q(I;\,X)$ stands for the set of measurable functions on I with values in X, such that $t\longmapsto\|f(t)\|_X$ belongs to $L^q(I)$. In particular, if $I=[0,T]$ for $T\in(0,+\infty)$, we denote $L^q([0,T];\,X)$ by $L^q_T(X)$ for short. Usually, we denote $\mathbb{P}$ the Leray projector over divergence-free vector fields defined as $\mathbb{P}\stackrel{\text{def}}{=}I+\nabla(-\Delta)^{-1}\text{div}$, where I means the identity operator.

2. Preliminaries

For the convenience of the reader, in what follows, we recall some basic facts on Littlewood-Paley theory, one may check [6, 25] for more details.

Lemma 2.1. *[Bernstein's inequality] Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^2$. A constant C exists so that for any positive real number λ, any non negative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers $(a,\ b)$ with $b\geq a\geq 1$, there hold*

$$\begin{aligned}
&\text{Supp }\ \hat{u}\subset\lambda\mathcal{B}\Rightarrow\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^b}\leq C^{k+1}\lambda^{k+2(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},\\
&\text{Supp }\ \hat{u}\subset\lambda\mathcal{C}\Rightarrow C^{-1-k}\lambda^k\|u\|_{L^a}\leq\sup_{|\alpha|=k}\|\partial^\alpha u\|_{L^a}\leq C^{1+k}\lambda^k\|u\|_{L^a},\\
&\text{Supp }\ \hat{u}\subset\lambda\mathcal{C}\Rightarrow\|\sigma(D)u\|_{L^b}\leq C_{\sigma,m}\lambda^{m+2(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}.
\end{aligned}\tag{2.1}$$

The proof of Theorem 1.1 requires a dyadic decomposition of the Fourier variables, which is called the Littlewood-Paley decomposition. Let us briefly explain how it may be built in the case $x\in\mathbb{R}^2$ (see e.g. [6]). Let φ be a smooth function supported in the ring $\mathcal{C}\stackrel{\text{def}}{=}\{\xi\in\mathbb{R}^2,\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ and $\chi(\xi)$ be a smooth function supported in the ball $\mathcal{B}\stackrel{\text{def}}{=}\{\xi\in\mathbb{R}^2,\ |\xi|\leq\frac{4}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1\quad\text{for}\quad\xi\neq0\quad\text{and}\quad\chi(\xi)+\sum_{q\geq0}\varphi(2^{-q}\xi)=1\quad\text{for all}\quad\xi\in\mathbb{R}^2.$$

Now, for $u\in\mathcal{S}'(\mathbb{R}^2)$, we set

$$\begin{aligned}
&\forall q\in\mathbb{Z},\quad\dot{\Delta}_q u=\varphi(2^{-q}\mathrm{D})u\qquad\text{and}\qquad\dot{S}_q u=\sum_{j\leq q-1}\dot{\Delta}_j u.\\
&q\geq0,\quad\Delta_q u=\varphi(2^{-q}\mathrm{D})u,\quad\Delta_{-1}u=\chi(\mathrm{D})u\quad\text{and}\quad S_q u=\sum_{-1\leq q'\leq q-1}\Delta_{q'}u.
\end{aligned}\tag{2.2}$$

We have the formal decomposition

$$u=\sum_{q\in\mathbb{Z}}\dot{\Delta}_q\,u,\quad\forall\,u\in\mathcal{S}'(\mathbb{R}^2)/\mathcal{P}[\mathbb{R}^2]\quad\text{and}\quad u=\sum_{q\geq-1}\Delta_q\,u,\quad\forall\,u\in\mathcal{S}'(\mathbb{R}^2),$$

where $\mathcal{P}[\mathbb{R}^2]$ is the set of polynomials (see [23]). Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\begin{aligned}
&\dot{\Delta}_k\dot{\Delta}_q u\equiv0\quad\text{if}\quad|k-q|\geq2\quad\text{and}\quad\dot{\Delta}_k(\dot{S}_{q-1}u\dot{\Delta}_q u)\equiv0\quad\text{if}\quad|k-q|\geq5,\\
&\Delta_k\Delta_q u\equiv0\quad\text{if}\quad|k-q|\geq2\quad\text{and}\quad\Delta_k(S_{q-1}u\Delta_q u)\equiv0\quad\text{if}\quad|k-q|\geq5.
\end{aligned}\tag{2.3}$$

We recall now the definitions of nonhomogeneous and homogeneous Besov spaces from [25, 6].

Definition 2.1 ([6]). *Let $s \in \mathbb{R}$, $1 \le p, r \le +\infty$, we set*

$$\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} \Big\| 2^{qs} \|\Delta_q u\|_{L^p} \Big\|_{\ell^r(\mathbb{N}\cup\{-1\})} \quad \text{and} \quad \|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \Big\| 2^{qs} \|\dot{\Delta}_q u\|_{L^p} \Big\|_{\ell^r(\mathbb{Z})}.$$

The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^2)$ consists of those distributions u in $\mathcal{S}'(\mathbb{R}^2)$ such that $\|u\|_{B^s_{p,r}} < \infty$, and the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^2)$ consists of those distributions u in $\mathcal{S}'_h(\mathbb{R}^2)$ such that $\|u\|_{\dot{B}^s_{p,r}} < \infty$, where $\mathcal{S}'_h(\mathbb{R}^2) \stackrel{\text{def}}{=} \{u \in \mathcal{S}'(\mathbb{R}^2) | \lim_{\lambda\to+\infty} \|\theta(\lambda D)u\|_{L^\infty} = 0$ for any $\theta \in \mathcal{D}(\mathbb{R}^2)\}$.

Remark 2.1. (1) *We point out that if $s > 0$ then $B^s_{p,r} = \dot{B}^s_{p,r} \cap L^p$ and*

$$\|u\|_{B^s_{p,r}} \approx \|u\|_{\dot{B}^s_{p,r}} + \|u\|_{L^p}.$$

(2) *It is easy to verify that the homogeneous Besov space $\dot{B}^s_{2,2}(\mathbb{R}^2)$ (resp. $B^s_{2,2}(\mathbb{R}^2)$) coincides with the classical homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^2)$ (resp. $H^s(\mathbb{R}^2)$) and $\dot{B}^s_{\infty,\infty}(\mathbb{R}^2)$ coincides with the classical homogeneous Hölder space $\dot{C}^s(\mathbb{R}^2)$ when s is not positive integer, in case s is a nonnegative integer, $\dot{B}^s_{\infty,\infty}(\mathbb{R}^2)$ coincides with the classical homogeneous Zygmund space $\dot{C}^s_*(\mathbb{R}^3)$.*

(3) *Let $s \in \mathbb{R}, 1 \le p,\, r \le +\infty$, and $u \in \mathcal{S}'_h(\mathbb{R}^2)$. Then u belongs to $\dot{B}^s_{p,r}(\mathbb{R}^2)$ if and only if there exists some positive constant C and some nonnegative sequence $\{c_{j,r}\}_{j\in\mathbb{Z}}$ such that $\|\{c_{j,r}\}\|_{\ell^r} = 1$ and $\forall\ j \in \mathbb{Z}$*

$$\|\dot{\Delta}_j u\|_{L^p} \le C c_{j,r}\, 2^{-js} \|u\|_{\dot{B}^s_{p,r}}. \tag{2.4}$$

Similarly, for $u \in \mathcal{S}'(\mathbb{R}^2)$, u belongs to $B^s_{p,r}(\mathbb{R}^2)$ if and only if there holds

$$\|\Delta_j u\|_{L^p} \le C c_{j,r}\, 2^{-js} \|u\|_{\dot{B}^s_{p,r}}. \tag{2.5}$$

Proposition 2.1 ([6]). *The following properties hold.*

(1) *For $s \in \mathbb{R}$, $1 \le p,\, r \le +\infty$, the normed space $(B^s_{p,r}(\mathbb{R}^2), \|\cdot\|_{B^s_{p,r}})$ is complete. When $s < \frac{2}{p}$ and $1 \le r \le +\infty$, or $s \le \frac{2}{p}$ and $r = 1$, the normed space $(\dot{B}^s_{p,r}(\mathbb{R}^2), \|\cdot\|_{\dot{B}^s_{p,r}})$ is complete.*

(2) *Sobolev embeddings: if $1 \le p_1 \le p_2 \le +\infty$, $1 \le r_1 \le r_2 \le +\infty$, $s \in \mathbb{R}$, and $s_1 < s_2$, then we have $\dot{B}^s_{p_1,r_1}(\mathbb{R}^2) \hookrightarrow \dot{B}^{s-2(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^2)$, $B^s_{p_1,r_1}(\mathbb{R}^2) \hookrightarrow B^{s-2(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}(\mathbb{R}^2)$, and $B^{s_2}_{p_1,r_1} \hookrightarrow B^{s_1}_{p_1,r_1}$.*

(3) *A constant C exists which satisfies the following properties. If s_1 and s_2 are real numbers such that $s_1 < s_2$, $\theta \in (0,1)$, $1 \le p,\, r \le +\infty$, then we have*

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \le \|u\|^\theta_{B^{s_1}_{p,r}} \|u\|^{1-\theta}_{B^{s_2}_{p,r}} \quad \text{and} \tag{2.6}$$

$$\|u\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} \le \frac{C}{s_2 - s_1}\Big(\frac{1}{\theta} + \frac{1}{1-\theta}\Big) \|u\|^\theta_{B^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}. \tag{2.7}$$

These assertions are true for homogeneous Besov spaces $\dot{B}^s_{p,r}$.

(4) *Let $m \in \mathbb{R}$ and f be a S^m-multiplier (that is, $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and satisfies that for all multi-index s, there exists a constant C_s such that $\forall \xi \in \mathbb{R}^d$, $|\partial^s f(\xi)| \le C_s(1 + |\xi|)^{m-|s|}$.) Then for all $s \in \mathbb{R}$ and $1 \le p,\, r \le +\infty$, the operator $f(D)$ in continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

In the rest of the paper, we shall frequently use Bony's decomposition [8] in both the homogeneous and the inhomogeneous context. The homogeneous Bony's decomposition reads

$$uv = T_u v + T'_v u = T_u v + T_v u + R(u, v), \tag{2.8}$$

where

$$T_u v \stackrel{\text{def}}{=} \sum_{q\in\mathbb{Z}} \dot{S}_{q-1}u\dot{\Delta}_q v,\ T'_v u \stackrel{\text{def}}{=} \sum_{q\in\mathbb{Z}} \dot{\Delta}_q u\dot{S}_{q+2}v,\ R(u,v) \stackrel{\text{def}}{=} \sum_{q\in\mathbb{Z}} \dot{\Delta}_q u\widetilde{\dot{\Delta}}_q v \text{ with } \widetilde{\dot{\Delta}}_q v \stackrel{\text{def}}{=} \sum_{|q'-q|\le 1} \dot{\Delta}_{q'} v,$$

and the inhomogeneous Bony's decomposition can be defined in a similar manner.

The main continuity properties of the paraproduct are described below, which provides us with product laws in inhomogeneous Besov spaces. These assertions hold also true in the homogeneous Besov spaces.

Proposition 2.2 ([6])**.** (1) *A constant C exists which satisfies the following inequalities. For $s\in\mathbb{R}$, $t<0$, $1\le p,\ r,\ r_1,\ r_2\le +\infty$, we have*

$$\|T_u v\|_{B^s_{p,r}} \le C^{|s|+1}\|u\|_{L^\infty}\|v\|_{B^s_{p,r}},$$
$$\|T_u v\|_{B^{s+t}_{p,r}} \le \frac{C^{|s+t|+1}}{-t}\|u\|_{B^t_{\infty,r_1}}\|v\|_{B^s_{p,r_2}} \quad \text{with} \quad \frac{1}{r} \stackrel{def}{=} \min\{1, \frac{1}{r_1}+\frac{1}{r_2}\}.$$

(2) *A constant C exists which satisfies the following inequalities. Let (s_1,s_2) be in $\mathbb{R}^2$ and (p_1,p_2,r_1,r_2) be in $[1,\infty]^4$. Assume that $\frac{1}{p}\stackrel{def}{=}\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{r}\stackrel{def}{=}\frac{1}{r_1}+\frac{1}{r_2}\le 1$. Then we have*

$$\|R(u,v)\|_{B^{s_1+s_2}_{p,r}} \le \frac{C^{s_1+s_2+1}}{s_1+s_2}\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}} \quad \text{if} \quad s_1+s_2>0,$$
$$\|R(u,v)\|_{B^0_{p,\infty}} \le C\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}} \quad \text{if} \quad r=1 \text{ and } s_1+s_2=0.$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use Chemin-Lerner type spaces from [9, 10].

Definition 2.2. *Let $s\in\mathbb{R}$, $1\le r,\lambda,p\le+\infty$, and $T>0$. we set*

$$\|u\|_{\widetilde{L}^\lambda_T(B^s_{p,r})} \stackrel{def}{=} \Big\|2^{qs}\|\Delta_q u\|_{L^\lambda_T(L^p)}\Big\|_{\ell^r(\mathbb{N}\cup\{-1\})} \quad \text{and} \quad \|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \stackrel{def}{=} \Big\|2^{qs}\|\dot{\Delta}_q u\|_{L^\lambda_T(L^p)}\Big\|_{\ell^r(\mathbb{Z})}.$$

For $s\in\mathbb{R}$, we define $\widetilde{L}^\lambda_T(B^s_{p,r}) \stackrel{def}{=} \big\{u\in\mathcal{S}'((0,T)\times\mathbb{R}^2)\ \big|\ \|u\|_{\widetilde{L}^\lambda_T(B^s_{p,r})}<\infty\big\}$ and $\widetilde{L}^\lambda_T(\dot{B}^s_{p,r}) \stackrel{def}{=} \big\{u\in\mathcal{S}'((0,T)\times\mathbb{R}^2)\ \big|\ \|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})}<\infty,\ \lim_{j\to-\infty}\dot{S}_j u=0 \text{ in } L^\lambda_T(L^\infty(\mathbb{R}^2))\big\}$.

In the particular case when $p=r=2$, we denote $\widetilde{L}^\lambda_T(B^s_{2,2})$ (resp. $\widetilde{L}^\lambda_T(\dot{B}^s_{2,2})$) by $\widetilde{L}^\lambda_T(H^s)$ (resp. $\widetilde{L}^\lambda_T(\dot{H}^s)$).

Remark 2.2. *It is easy to observe that for $\theta\in[0,1]$, we have*

$$\|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \le \|u\|^\theta_{\widetilde{L}^{\lambda_1}_T(\dot{B}^{s_1}_{p,r})}\|u\|^{1-\theta}_{\widetilde{L}^{\lambda_2}_T(\dot{B}^{s_2}_{p,r})} \tag{2.9}$$

with $\frac{1}{\lambda}=\frac{\theta}{\lambda_1}+\frac{1-\theta}{\lambda_2}$ and $s=\theta s_1+(1-\theta)s_2$. Moreover, Minkowski inequality implies that

$$\|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \le \|u\|_{L^\lambda_T(\dot{B}^s_{p,r})} \quad \text{if} \quad \lambda\le r \quad \text{and} \quad \|u\|_{L^\lambda_T(\dot{B}^s_{p,r})} \le \|u\|_{\widetilde{L}^\lambda_T(\dot{B}^s_{p,r})} \quad \text{if} \quad r\le\lambda.$$

Let's now recall the following commutator's estimate which will be frequently used throughout the succeeding sections.

Lemma 2.2 (Commutator estimate, Lemma 1 in [24], Lemma 2.97 in [6])**.** *Let $(p,q,r)\in[1,\infty]^3$, θ be a C^1 function on $\mathbb{R}^d$ such that $(1+|\cdot|)\hat{\theta}\in L^1$. There exists a constant C such that for any Lipschitz function a with gradient in L^p and any function b in L^q, we have, for any positive λ,*

$$\|[\theta(\lambda^{-1}D),a]b\|_{L^r} \le C\lambda^{-1}\|\nabla a\|_{L^p}\|b\|_{L^q} \quad \text{with} \quad p^{-1}+q^{-1}=r^{-1}. \tag{2.10}$$

Almost all the estimates of the nonlinear terms in the paper are based on the following commutator estimates, which proofs rely on Bony's decomposition (2.8) and (2.10). Although their proofs are similar to the ones used in previous papers, some new observations and their applications play a crucial role to deal with the functions in critical spaces. So we prove them for the sake of self-containedness.

Lemma 2.3. (1) *Let $\alpha \in (-1,1)$, $(p,r) \in [1,\infty]^2$, $u \in \dot{B}^{\alpha}_{p,r}(\mathbb{R}^2)$ and $\nabla v \in L^\infty(\mathbb{R}^2)$ with* $\operatorname{div} v = 0$. *Then $\forall\, q \in \mathbb{Z}$, there holds*

$$\|[\dot{\Delta}_q, v\cdot\nabla]u\|_{L^p} \lesssim c_{q,r}2^{-q\alpha}\|\nabla v\|_{L^\infty}\|u\|_{\dot{B}^{\alpha}_{p,r}}. \tag{2.11}$$

(2) *Let $u \in \dot{B}^1_{2,1}(\mathbb{R}^2)$ and $\nabla v \in \dot{B}^0_{\infty,1}(\mathbb{R}^2)$ with* $\operatorname{div} v = 0$. *Then $\forall\, q \in \mathbb{Z}$, there holds*

$$\|[\dot{\Delta}_q, v\cdot\nabla]u\|_{L^2} \lesssim c_{q,1}2^{-q}\|\nabla v\|_{\dot{B}^0_{\infty,1}}\|u\|_{\dot{B}^1_{2,1}}. \tag{2.12}$$

(3) *Let $r = 1$ or 2, $u \in \dot{B}^1_{2,r}(\mathbb{R}^2)$ with* $\operatorname{div} u = 0$. *Then $\forall\, q \in \mathbb{Z}$, there holds*

$$\|[\dot{\Delta}_q, u\cdot\nabla]u\|_{L^2} \lesssim c_{q,r}\|\nabla u\|_{L^2}\|u\|_{\dot{B}^1_{2,r}}. \tag{2.13}$$

(4) *Let $u \in B^{-1}_{2,\infty}(\mathbb{R}^2)$ and $v \in B^1_{\infty,1}(\mathbb{R}^2)$ with* $\operatorname{div} v = 0$. *Then $\forall\, q \in \mathbb{N}\cup\{-1\}$, there holds*

$$\|[\Delta_q, v\cdot\nabla]u\|_{L^2} \lesssim c_{q,\infty}2^{q}\|v\|_{B^1_{\infty,1}}\|u\|_{B^{-1}_{2,\infty}}. \tag{2.14}$$

(5) *Let $u \in B^0_{2,1}(\mathbb{R}^2)$, $a \in B^1_{\infty,\infty}(\mathbb{R}^2)$ and $\nabla a \in L^\infty(\mathbb{R}^2)$. Then $\forall\, q \in \mathbb{N}\cup\{-1\}$, there holds*

$$\|[a, \Delta_q]\nabla u\|_{L^2} \lesssim c_{q,\infty}\big(\|\nabla a\|_{L^\infty} + \|a\|_{B^1_{\infty,\infty}}\big)\|u\|_{B^0_{2,1}}. \tag{2.15}$$

(6) *Let $\nabla u \in H^{\frac12}(\mathbb{R}^2)$ and $a \in B^1_{\infty,1}(\mathbb{R}^2)$. Then $\forall\, q \in \mathbb{N}\cup\{-1\}$, there holds*

$$\big\|[a, \Delta_q]\nabla u\big\|_{L^2} \lesssim c_{q,1}2^{-q}(\|a\|_{B^1_{\infty,1}}\|\nabla u\|_{L^2} + \|a\|_{B^{\frac12}_{\infty,1}}\|\nabla u\|_{H^{\frac12}}). \tag{2.16}$$

(7) *Let $\varepsilon \in (0,1)$, $f \in L^2(\mathbb{R}^2)$ and $a \in \dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}(\mathbb{R}^2)$. Then $\forall\, q \in \mathbb{Z}$, there holds*

$$\big\|[a, \dot{\Delta}_q]f\big\|_{L^2} \lesssim c_{q,1}\|a\|_{\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{L^2} \tag{2.17}$$

(8) *Let $\varepsilon \in (0,1)$, $f \in \dot{B}^{-1}_{2,1}(\mathbb{R}^2)$ and $a \in \dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1}(\mathbb{R}^2)$. Then $\forall\, q \in \mathbb{Z}$, there holds*

$$\big\|[a, \dot{\Delta}_q]f\big\|_{L^2} \lesssim c_{q,1}(\|\nabla a\|_{L^\infty}\|f\|_{\dot{B}^{-1}_{2,1}} + \|a\|_{\dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{\dot{B}^{-1}_{2,2}}) \tag{2.18}$$

Proof. Thanks to Bony's decomposition (2.8), we have

$$\begin{aligned}[\dot{\Delta}_q, v\cdot\nabla]u &= \dot{\Delta}_q(v\cdot\nabla u) - v\cdot\nabla\dot{\Delta}_q u\\ &= \dot{\Delta}_q(T_{v^j}\partial_j u) + \dot{\Delta}_q(T_{\partial_j u}v^j) + \dot{\Delta}_q R(v^j,\partial_j u) - (T_{v^j}\dot{\Delta}_q\partial_j u + T'_{\dot{\Delta}_q\partial_j u}v^j),\end{aligned}$$

which along with the divergence free condition of v yields

$$[\dot{\Delta}_q, v\cdot\nabla]u = \dot{\Delta}_q\big(\partial_j R(v^j,u)\big) + \dot{\Delta}_q\big(T_{\partial_j u}v^j\big) - T'_{\dot{\Delta}_q\partial_j u}v^j + [\dot{\Delta}_q, T_{v^j}]\partial_j u \overset{\text{def}}{=} \sum_{i=1}^4 \mathcal{R}^i_q. \tag{2.19}$$

For $\mathcal{R}^1_q = \dot{\Delta}_q\big(\partial_j R(v^j,u)\big) = \sum_{k\geq q-3}\dot{\Delta}_q\partial_j(\dot{\Delta}_k v^j\widetilde{\dot{\Delta}}_k u)$, it follows from Lemma 2.1 that

$$\|\mathcal{R}^1_q\|_{L^p} \lesssim 2^q\sum_{k\geq q-3}\|\dot{\Delta}_k v^j\widetilde{\dot{\Delta}}_k u\|_{L^p} \lesssim 2^q\sum_{k\geq q-3}\|\widetilde{\dot{\Delta}}_k u\|_{L^p}\|\dot{\Delta}_k v^j\|_{L^\infty}. \tag{2.20}$$

Hence, we obtain from (2.4) that, for $\alpha > -1$,

$$\|\mathcal{R}_q^1\|_{L^p} \lesssim 2^q \|\nabla v\|_{L^\infty} \sum_{k\geq q-3} \|\widetilde{\dot{\Delta}}_k u\|_{L^p} 2^{-k} \lesssim c_{q,r} 2^{-q\alpha} \|\nabla v\|_{L^\infty} \|u\|_{\dot{B}^\alpha_{p,r}}.$$

Notice that for $\alpha < 1$, we get from (2.4) that

$$\|\dot{S}_{k-1}\nabla u\|_{L^p} \lesssim \sum_{\ell\leq k-2} 2^\ell \|\dot{\Delta}_\ell u\|_{L^p} \lesssim c_{k,r} 2^{k(1-\alpha)} \|u\|_{\dot{B}^\alpha_{p,r}},$$

then for $\mathcal{R}_q^2 = \dot{\Delta}_q\big(T_{\partial_j u} v^j\big) = \sum_{|q-k|\leq 4} \dot{\Delta}_q(\dot{S}_{k-1}\partial_j u \dot{\Delta}_k v^j)$, we obtain

$$\|\mathcal{R}_q^2\|_{L^p} \lesssim \sum_{|q-k|\leq 4} \|\dot{S}_{k-1}\partial_j u\|_{L^p} \|\dot{\Delta}_k v^j\|_{L^\infty} \lesssim c_{q,r} 2^{-q\alpha} \|\nabla v\|_{L^\infty} \|u\|_{\dot{B}^\alpha_{p,r}}. \tag{2.21}$$

Whereas for $\alpha = 1$, we deduce from $\|\dot{S}_{k-1}\nabla u\|_{L^2} \lesssim \sum_{\ell\leq k-2} 2^\ell \|\dot{\Delta}_\ell u\|_{L^2} \lesssim \|u\|_{\dot{B}^1_{2,1}}$ that

$$\|\mathcal{R}_q^2\|_{L^2} \lesssim \sum_{|q-k|\leq 4} \|\dot{S}_{k-1}\partial_j u\|_{L^2} \|\dot{\Delta}_k v^j\|_{L^\infty} \lesssim c_{q,1} 2^{-q} \|\nabla v\|_{\dot{B}^0_{\infty,1}} \|u\|_{\dot{B}^1_{2,1}}.$$

Thanks to the properties to the support of Fourier transform to $\dot{S}_{k+2}\dot{\Delta}_q\partial_j u$, for $\mathcal{R}_q^3 = -T'_{\dot{\Delta}_q\partial_j u} v^j = -\sum_{k\geq q-3} \dot{S}_{k+2}\dot{\Delta}_q\partial_j u \dot{\Delta}_k v^j$, one obtains that for $\alpha \in \mathbb{R}$

$$\begin{aligned}\|\mathcal{R}_q^3\|_{L^p} &\lesssim \|\dot{\Delta}_q\partial_j u\|_{L^p} \sum_{k\geq q-3} \|\dot{\Delta}_k v^j\|_{L^\infty} \lesssim \|\dot{\Delta}_q u\|_{L^p} \sum_{k\geq q-3} 2^{q-k}\, \|\dot{\Delta}_k \nabla v\|_{L^\infty} \\ &\lesssim c_{q,r} 2^{-q\alpha} \|u\|_{\dot{B}^\alpha_{p,r}} \|\nabla v\|_{L^\infty}.\end{aligned}$$

For the last term in (2.19), owing to the properties of spectral localization of the Littlewood-Paley decomposition, we have

$$\mathcal{R}_q^4 = [\dot{\Delta}_q, T_{v^j}]\partial_j u = \sum_{|k-q|\leq 4} [\dot{\Delta}_q, \dot{S}_{k-1} v^j] \dot{\Delta}_k \partial_j u.$$

Applying Lemma 2.2 leads to

$$\|\dot{\Delta}_q, \dot{S}_{k-1} v^j]\dot{\Delta}_k\partial_j u\|_{L^p} \lesssim 2^{-q} \|\nabla \dot{S}_{k-1} v\|_{L^\infty} \|\partial_j \dot{\Delta}_k u\|_{L^p} \lesssim 2^{k-q} \|\nabla v\|_{L^\infty} \|\dot{\Delta}_k u\|_{L^p}.$$

We thus get, for $\alpha \in \mathbb{R}$,

$$\begin{aligned}\|\mathcal{R}_q^4\|_{L^p} &\lesssim \sum_{|k-q|\leq 4} \|[\dot{\Delta}_q, \dot{S}_{k-1} v^j]\dot{\Delta}_k\partial_j u\|_{L^p} \\ &\lesssim \|\nabla v\|_{L^\infty} \sum_{|k-q|\leq 4} \|\dot{\Delta}_k u\|_{L^p} \lesssim c_{q,r} 2^{-q\alpha} \|u\|_{\dot{B}^\alpha_{p,r}} \|\nabla v\|_{L^\infty}.\end{aligned} \tag{2.22}$$

Therefore, owing to $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, we arrive at (2.11) and (2.12).

Similarly, in order to prove (2.13), we shall use the decomposition (2.19) with $v = u$. We first apply Lemma 2.1 to $\mathcal{R}_q^1 = \sum_{k\geq q-3} \dot{\Delta}_q\partial_j(\dot{\Delta}_k u^j \widetilde{\dot{\Delta}}_k u)$ to get

$$\|\mathcal{R}_q^1\|_{L^2} \lesssim 2^{2q} \sum_{k\geq q-3} \|\dot{\Delta}_k u^j \widetilde{\dot{\Delta}}_k u\|_{L^1} \lesssim 2^{2q} \sum_{k\geq q-3} \|\widetilde{\dot{\Delta}}_k u\|_{L^2} \|\dot{\Delta}_k u\|_{L^2},$$

which follows

$$\|\mathcal{R}_q^1\|_{L^2} \lesssim 2^{2q} \|\nabla u\|_{L^2} \|u\|_{\dot{B}^1_{2,r}} \sum_{k\geq q-3} 2^{-2k} c_{k,2} c_{k,r} \lesssim c_{q,1} \|\nabla u\|_{L^2} \|u\|_{\dot{B}^1_{2,r}}.$$

Thanks to Lemma 2.1 again, one has

$$\begin{aligned}\|\mathcal{R}_q^2\|_{L^2} &\lesssim 2^q \sum_{|q-k|\leq 4} \|\dot{S}_{k-1}\partial_j u \dot{\Delta}_k u^j\|_{L^1}\\ &\lesssim 2^q \sum_{|q-k|\leq 4} \|\dot{S}_{k-1}\partial_j u\|_{L^2}\|\dot{\Delta}_k u^j\|_{L^2} \lesssim c_{q,r}\|\nabla u\|_{L^2}\|u\|_{\dot{B}^1_{2,r}}.\end{aligned}$$

For $\mathcal{R}_q^3 = -\sum_{k\geq q-3}\dot{S}_{k+2}\dot{\Delta}_q\partial_j u\dot{\Delta}_k u^j$, we find

$$\begin{aligned}\|\mathcal{R}_q^3\|_{L^2} &\lesssim \|\dot{\Delta}_q\partial_j u\|_{L^\infty}\sum_{k\geq q-3}\|\dot{\Delta}_k u^j\|_{L^2} \lesssim 2^q\|\dot{\Delta}_q\nabla u\|_{L^2}\|u\|_{\dot{B}^1_{2,r}}\sum_{k\geq q-3}2^{-k}c_{k,r}\\ &\lesssim c_{q,2}\|\nabla u\|_{L^2}\|u\|_{\dot{B}^1_{2,r}}c_{q,r} \lesssim c_{q,1}\|\nabla u\|_{L^2}\|u\|_{\dot{B}^1_{2,r}}.\end{aligned}$$

While for $\mathcal{R}_q^4 = [\dot{\Delta}_q, T_{v^j}]\partial_j u = \sum_{|k-q|\leq 4}[\dot{\Delta}_q, \dot{S}_{k-1}v^j]\dot{\Delta}_k\partial_j u$, we deduce from Lemma 2.2 that

$$\|\dot{\Delta}_q, \dot{S}_{k-1}u^j]\dot{\Delta}_k\partial_j u\|_{L^2} \lesssim 2^{-q}\|\nabla\dot{S}_{k-1}u\|_{L^\infty}\|\partial_j\dot{\Delta}_k u\|_{L^2}.$$

Remark that

$$\|\nabla\dot{S}_{k-1}u\|_{L^\infty} \lesssim \sum_{\ell\leq k-2}\|\nabla\dot{\Delta}_\ell u\|_{L^\infty} \lesssim \sum_{\ell\leq k-2}2^\ell\|\nabla\dot{\Delta}_\ell u\|_{L^2} \lesssim \|\nabla u\|_{L^2}2^k c_{k,2},$$

we thus get

$$\|\mathcal{R}_q^4\|_{L^2} \lesssim \|u\|_{\dot{B}^1_{2,r}}\|\nabla u\|_{L^2}2^{-q}\sum_{|k-q|\leq 4}2^k c_{k,2}c_{k,r} \lesssim c_{q,1}\|\nabla u\|_{L^2}\|u\|_{\dot{B}^1_{2,r}}.$$

Hence, we prove (2.13).

Next, due to the inhomogeneous Bony's decomposition (2.8), we may find the inhomogeneous version of (2.19). In this case, similar to (2.20), one can see

$$\begin{aligned}\|\mathcal{R}_q^1\|_{L^2} &\lesssim 2^q\sum_{k\geq q-3}\|\widetilde{\Delta}_k u\|_{L^2}\|\Delta_k v^j\|_{L^\infty}\\ &\lesssim \|u\|_{B^{-1}_{2,\infty}}\|v\|_{B^1_{\infty,1}}2^q\sum_{k\geq q-3}c_{k,\infty}c_{k,1} \lesssim c_{q,\infty}2^q\|v\|_{B^1_{\infty,1}}\|u\|_{B^{-1}_{2,\infty}}.\end{aligned}$$

For $\mathcal{R}_q^2$, we get from the proof of (2.21) that

$$\|\mathcal{R}_q^2\|_{L^2} \lesssim c_{q,\infty}2^q\|\nabla v\|_{L^\infty}\|u\|_{B^{-1}_{2,\infty}} \lesssim c_{q,\infty}2^q\|v\|_{B^1_{\infty,1}}\|u\|_{B^{-1}_{2,\infty}}.$$

And for $\mathcal{R}_q^3 = -\sum_{k\geq q-3}S_{k+2}\Delta_q\partial_j u\Delta_k v^j$, we prove

$$\begin{aligned}\|\mathcal{R}_q^3\|_{L^2} &\lesssim \|\Delta_q\partial_j u\|_{L^2}\sum_{k\geq q-3}\|\Delta_k v^j\|_{L^\infty} \lesssim 2^q c_{q,\infty}\|u\|_{B^{-1}_{2,\infty}}\sum_{k\geq q-3}2^{q-k}\,c_{k,1}\|v\|_{B^1_{\infty,1}}\\ &\lesssim c_{q,\infty}2^q\|v\|_{B^1_{\infty,1}}\|u\|_{B^{-1}_{2,\infty}}.\end{aligned}$$

Finally, similar to the proof of (2.22), we obtain

$$\|\mathcal{R}_q^4\|_{L^2} \lesssim c_{q,\infty}2^q\|v\|_{B^1_{\infty,1}}\|u\|_{B^{-1}_{2,\infty}}.$$

Hence, we prove (2.14).

Thanks to Bony's decomposition (2.8) in the inhomogeneous context again, we write

$$[a,\Delta_q]\nabla u = \Delta_q\,R(a,\nabla u) + \Delta_q\,T_{\nabla u}a - T'_{\Delta_q\nabla u}a - [T_a,\Delta_q]\nabla u. \tag{2.23}$$

Whereas applying Lemma 2.1 gives

$$\|\Delta_q R(a,\nabla u)\|_{L^2} \lesssim \sum_{k\geq q-3}\|\Delta_k a\|_{L^\infty}\|\widetilde{\Delta}_k\nabla u\|_{L^2}, \tag{2.24}$$

which follows that

$$\begin{aligned}\|\Delta_q R(a,\nabla u)\|_{L^2} &\lesssim \sum_{k\geq q-3} 2^k\|\Delta_k a\|_{L^\infty}\|\widetilde{\Delta}_k u\|_{L^2}\\ &\lesssim \|a\|_{B^1_{\infty,\infty}}\|u\|_{B^0_{2,1}}\sum_{k\geq q-3} c_{k,1} \lesssim c_{q,\infty}\|a\|_{B^1_{\infty,\infty}}\|u\|_{B^0_{2,1}}.\end{aligned}$$

In a similar manner, for $T'_{\Delta_q\nabla u}a=\sum_{k\geq q-3}S_{k+2}\Delta_q\nabla u\Delta_k a$, we prove

$$\begin{aligned}\|T'_{\Delta_q\nabla u}a\|_{L^2} &\lesssim \sum_{k\geq q-3}\|S_{k+2}\Delta_q\nabla u\|_{L^2}\|\Delta_k a\|_{L^\infty} \lesssim \|\Delta_q\nabla u\|_{L^2}\sum_{k\geq q-3}\|\Delta_k a\|_{L^\infty}\\ &\lesssim \|a\|_{B^1_{\infty,\infty}}\|u\|_{B^0_{2,1}}2^q c_{q,1}\sum_{k\geq q-3}2^{-k}c_{k,\infty} \lesssim c_{q,1}\|a\|_{B^1_{\infty,\infty}}\|u\|_{B^0_{2,1}}.\end{aligned}$$

Due to Lemma 2.1, we have

$$\|S_{k-1}\nabla u\|_{L^2} \lesssim \sum_{\ell\leq k-2}2^\ell\|\Delta_\ell u\|_{L^2} \lesssim \|u\|_{B^0_{2,1}}\sum_{\ell\leq k-2}2^\ell c_{\ell,1} \lesssim \|u\|_{B^0_{2,1}}2^k c_{k,1},$$

which leads to

$$\begin{aligned}\|\Delta_q T_{\nabla u}a(t)\|_{L^2} &\lesssim \sum_{|q-k|\leq 4}\|\Delta_k a\|_{L^\infty}\|S_{k-1}\nabla u\|_{L^2}\\ &\lesssim \|a\|_{B^1_{\infty,\infty}}\|u\|_{B^0_{2,1}}\sum_{|q-k|\leq 4}c_{k,\infty}c_{k,1} \lesssim c_{q,1}\|a\|_{B^1_{\infty,\infty}}\|u\|_{B^0_{2,1}}.\end{aligned}$$

Since $\|\nabla S_{k-1}a\|_{L^\infty}\lesssim\|\nabla a\|_{L^\infty}$, we deduce from Lemma 2.2 that

$$\|[\Delta_q,T_a]\nabla u\|_{L^2} \lesssim \sum_{|k-q|\leq 4}2^{-q}\|\nabla S_{k-1}a\|_{L^\infty}\|\nabla\Delta_k u\|_{L^2} \lesssim c_{q,1}\|\nabla a\|_{L^\infty}\|u\|_{B^0_{2,1}}.$$

As a consequence, we obtain (2.15).

In order to prove (2.16), we will use the decomposition (2.23) of the term $\big[a,\Delta_q\big]\nabla u$ again. We first apply (2.24) to get

$$\|\Delta_q R(a,\nabla u)\|_{L^2} \lesssim \|a\|_{B^1_{\infty,1}}\|\nabla u\|_{L^2}\sum_{k\geq q-3}2^{-k}c_{k,1}c_{k,2} \lesssim 2^{-q}c_{q,1}\|a\|_{B^1_{\infty,1}}\|\nabla u\|_{L^2}.$$

For $T'_{\Delta_q\nabla u}a=\sum_{k\geq q-3}S_{k+2}\Delta_q\nabla u\Delta_k a$, we prove

$$\begin{aligned}\|T'_{\Delta_q\nabla u}a\|_{L^2} &\lesssim \sum_{k\geq q-3}\|S_{k+2}\Delta_q\nabla u\|_{L^2}\|\Delta_k a\|_{L^\infty} \lesssim \|\nabla u\|_{L^2}\sum_{k\geq q-3}\|\Delta_k a\|_{L^\infty}\\ &\lesssim \|\nabla u\|_{L^2}\|a\|_{B^1_{\infty,1}}\sum_{k\geq q-3}2^{-k}c_{k,1} \lesssim 2^{-q}c_{q,1}\|a\|_{B^1_{\infty,1}}\|\nabla u\|_{L^2}.\end{aligned}$$

Note that $\|S_{k-1}\nabla u\|_{L^2}\lesssim\|\nabla u\|_{L^2}$, it follows from Lemma 2.1 that

$$\begin{aligned}\|\Delta_q T_{\nabla u}a(t)\|_{L^2} &\lesssim \sum_{|q-k|\leq 4}\|\Delta_k a\|_{L^\infty}\|S_{k-1}\nabla u\|_{L^2}\\ &\lesssim \|a\|_{B^1_{\infty,1}}\|\nabla u\|_{L^2}\sum_{|q-k|\leq 4}2^{-k}c_{k,1} \lesssim c_{q,1}2^{-q}\|a\|_{B^1_{\infty,1}}\|\nabla u\|_{L^2}.\end{aligned}$$

Finally, thanks to

$$\|\nabla S_{k-1}a\|_{L^\infty} \lesssim \sum_{\ell\leq k-2}2^{2\ell}\|\Delta_\ell a\|_{L^2} \lesssim \|a\|_{B^{\frac12}_{\infty,1}}\sum_{\ell\leq k-2}c_{\ell,1}2^{\frac12\ell} \lesssim \|a\|_{B^{\frac12}_{\infty,1}}c_{k,1}2^{\frac12 k},$$

we obtain from Lemma 2.2 that

$$\begin{aligned}\|[\Delta_q, T_a]\nabla u\|_{L^2} &\lesssim \sum_{|k-q|\leq 4} 2^{-q}\|\nabla S_{k-1}a\|_{L^\infty}\|\nabla\Delta_k u\|_{L^2}\\ &\lesssim \|a\|_{B^{\frac12}_{\infty,1}}\|\nabla u\|_{H^{\frac12}}\sum_{|k-q|\leq 4} 2^{-q}c_{k,1}c_{k,2} \lesssim 2^{-q}c_{q,1}\|a\|_{B^{\frac12}_{\infty,1}}\|\nabla u\|_{H^{\frac12}}.\end{aligned}$$

Therefore, we get (2.16).

Toward (2.17), we will use the homogeneous version of the decomposition (2.23), that is,

$$[a,\dot\Delta_q]f = \dot\Delta_q\, R(a,f) + \dot\Delta_q\, T_f a - T'_{\dot\Delta_q f}a - [T_a,\dot\Delta_q]f. \tag{2.25}$$

Thanks to Lemma 2.1, we find

$$\|\dot\Delta_q R(a,f)\|_{L^2} \lesssim 2^{q\varepsilon}\sum_{k\geq q-3}\|\dot\Delta_k a\widetilde{\dot\Delta}_k f\|_{L^{\frac{2}{1+\varepsilon}}} \lesssim 2^{q\varepsilon}\sum_{k\geq q-3}\|\dot\Delta_k a\|_{L^{\frac{2}{\varepsilon}}}\|\widetilde{\dot\Delta}_k f\|_{L^2},$$

which implies

$$\|\dot\Delta_q R(a,f)\|_{L^2} \lesssim 2^{q\varepsilon}\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{L^2}\sum_{k\geq q-3} c_{k,1}2^{-k\varepsilon}c_{k,2} \lesssim c_{q,1}\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{L^2}.$$

In a similar manner, for $T'_{\dot\Delta_q f}a = \sum_{k\geq q-3}\dot S_{k+2}\dot\Delta_q f\dot\Delta_k a$, we prove

$$\|T'_{\dot\Delta_q f}a\|_{L^2} \lesssim 2^{q\varepsilon}\sum_{k\geq q-3}\|\dot S_{k+2}\dot\Delta_q f\dot\Delta_k a\|_{L^{\frac{2}{1+\varepsilon}}} \lesssim \|\dot\Delta_q f\|_{L^2}\sum_{k\geq q-3}\|\dot\Delta_k a\|_{L^{\frac{2}{\varepsilon}}}, \tag{2.26}$$

which follows

$$\|T'_{\dot\Delta_q f}a\|_{L^2} \lesssim 2^{q\varepsilon}\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{L^2}c_{q,2}\sum_{k\geq q-3} c_{k,1}2^{-k\varepsilon} \lesssim c_{q,1}\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{L^2}.$$

Due to Lemma 2.1, we deduce from $\|\dot S_{k-1}f\|_{L^2} \lesssim \|f\|_{L^2}$ that

$$\begin{aligned}\|\dot\Delta_q T_f a\|_{L^2} &\lesssim \sum_{|q-k|\leq 4}\|\dot\Delta_k a\|_{L^\infty}\|\dot S_{k-1}f\|_{L^2}\\ &\lesssim \|f\|_{L^2}\sum_{|q-k|\leq 4} 2^{k\varepsilon}\|\dot\Delta_k a\|_{L^{\frac{2}{\varepsilon}}} \lesssim c_{q,1}\|f\|_{L^2}\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}.\end{aligned}$$

Finally notice that

$$\|\nabla\dot S_{k-1}a\|_{L^\infty} \lesssim \sum_{\ell\leq k-2}\|\nabla\dot\Delta_\ell a\|_{L^\infty} \lesssim \sum_{\ell\leq k-2} 2^{\ell(1+\varepsilon)}\|\dot\Delta_k a\|_{L^{\frac{2}{\varepsilon}}} \lesssim \|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}\sum_{\ell\leq k-2} 2^\ell c_{\ell,1} \lesssim \|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}} 2^k c_{k,1}$$

one gets from Lemma 2.2 that

$$\begin{aligned}\|[\dot\Delta_q, T_a]f\|_{L^2} &\lesssim \sum_{|k-q|\leq 4} 2^{-q}\|\nabla\dot S_{k-1}a\|_{L^\infty}\|\dot\Delta_k f\|_{L^2}\\ &\lesssim \|f\|_{L^2}\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}\sum_{|k-q|\leq 4} 2^{k-q}c_{k,2}c_{k,1} \lesssim c_{q,1}\|f\|_{L^2}\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}.\end{aligned}$$

As a consequence, we obtain (2.17).

Along the same line as in the proof of (2.17), for the decomposition (2.25), we first deduce from Lemma 2.1 that

$$\|\dot\Delta_q\, R(a,f) + \dot\Delta_q\, T_f a\|_{L^2} \lesssim 2^{q\varepsilon}\sum_{k\geq q-3}\|\dot\Delta_k a\dot S_{k+2}f\|_{L^{\frac{2}{1+\varepsilon}}} \lesssim 2^{q\varepsilon}\sum_{k\geq q-3}\|\dot\Delta_k a\|_{L^{\frac{2}{\varepsilon}}}\|\dot S_{k+2}f\|_{L^2},$$

which along with $\|\dot{S}_{k+2}f\|_{L^2} \lesssim \sum_{\ell\le k+1}\|\dot{\Delta}_\ell f\|_{L^2} \lesssim \|f\|_{\dot{B}^{-1}_{2,2}} 2^k c_{k,2}$ implies

$$\|\dot{\Delta}_q\, R(a,f) + \dot{\Delta}_q\, T_f a\|_{L^2} \lesssim 2^{q\varepsilon}\|a\|_{\dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{\dot{B}^{-1}_{2,2}} \sum_{k\ge q-3} c_{k,1}2^{-k\varepsilon}c_{k,2} \lesssim c_{q,1}\|a\|_{\dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{\dot{B}^{-1}_{2,2}}.$$

Similarly, for $T'_{\dot{\Delta}_q f}a = \sum_{k\ge q-2}\dot{S}_{k+2}\dot{\Delta}_q f\,\dot{\Delta}_k a$, we obtain from (2.26) that

$$\|T'_{\dot{\Delta}_q f}a\|_{L^2} \lesssim 2^{q(1+\varepsilon)}\|a\|_{\dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{\dot{B}^{-1}_{2,2}}c_{q,2}\sum_{k\ge q-3} c_{k,1}2^{-k(1+\varepsilon)} \lesssim c_{q,1}\|a\|_{\dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1}}\|f\|_{\dot{B}^{-1}_{2,2}}.$$

For $[\dot{\Delta}_q, T_a]f$, thanks to Lemma 2.2 again, we prove

$$\begin{aligned}\|[\dot{\Delta}_q, T_a]f\|_{L^2} &\lesssim \sum_{|k-q|\le 4} 2^{-q}\|\nabla\dot{S}_{k-1}a\|_{L^\infty}\|\dot{\Delta}_k f\|_{L^2}\\ &\lesssim \|\nabla a\|_{L^\infty}\|f\|_{\dot{B}^{-1}_{2,1}}\sum_{|k-q|\le 4}2^{k-q}c_{k,1} \lesssim c_{q,1}\|\nabla a\|_{L^\infty}\|f\|_{\dot{B}^{-1}_{2,1}}.\end{aligned}$$

Hence, we obtain (2.18).

This achieves the proof of Lemma 2.3. □

Remark 2.3. *By virtue of* [24]*, we may obtain that all the assertions in Lemma 2.3 hold true if the dyadic operator Δ_q (or $\dot{\Delta}_q$) is replaced by $\sigma(D)\Delta_q$ (or $\sigma(D)\dot{\Delta}_q$) with any zero-order multiplier $\sigma(D)$.*

Applying Lemma 2.3 to the transport equation, we have

Proposition 2.3. *Let $k\in\mathbb{Z}$, $\alpha\in[0,1]$, $(p,r)\in[1,\infty]^2$, $a_0\in\dot{B}^\alpha_{p,r}(\mathbb{R}^2)$, $\nabla u\in L^1_T(\dot{B}^0_{\infty,1}(\mathbb{R}^2))$ with* $\operatorname{div} u = 0$*, and the function $a\in\mathcal{C}([0,T];\,\dot{B}^\alpha_{p,r}(\mathbb{R}^2)))$ solves*

$$\begin{cases}\partial_t a + u\cdot\nabla a = 0, \quad \forall\,(t,x)\in[0,T]\times\mathbb{R}^2,\\ a|_{t=0} = a_0, \quad \forall\,x\in\mathbb{R}^2.\end{cases} \tag{2.27}$$

Then there holds that for $\forall\, t\in(0,T]$

$$\|a-\dot{S}_k a\|_{\widetilde{L}^\infty_t(\dot{B}^\alpha_{p,r})} \le \Big(\sum_{q\ge k}2^{rq\alpha}\|\dot{\Delta}_q a_0\|^r_{L^p}\Big)^{\frac{1}{r}} + \|a_0\|_{\dot{B}^\alpha_{p,r}}(e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1) \quad \text{if} \quad \alpha\in[0,1), \tag{2.28}$$

and

$$\|a-\dot{S}_k a\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})} \le \sum_{q\ge k}2^q\|\dot{\Delta}_q a_0\|_{L^2} + \|a_0\|_{\dot{B}^1_{2,1}}(e^{C\|\nabla u\|_{L^1_t(\dot{B}^0_{\infty,1})}}-1). \tag{2.29}$$

Proof. Motivated by [2], we first apply $\dot{\Delta}_q$ to (2.27) to yield

$$\partial_t\dot{\Delta}_q a + u\cdot\nabla\dot{\Delta}_q a = [u\cdot\nabla, \dot{\Delta}_q]a$$

From the maximum principle we deduce

$$\|\dot{\Delta}_q a(t)\|_{L^p} \le \|\dot{\Delta}_q a_0\|_{L^p} + \int_0^t \|[u\cdot\nabla, \dot{\Delta}_q]a\|_{L^p}(\tau)\,d\tau. \tag{2.30}$$

For $\alpha\in[0,1)$, thanks to (2.11), we have

$$\|[u\cdot\nabla, \dot{\Delta}_q]a\|_{L^p} \lesssim c_{q,r}2^{-q\alpha}\|\nabla u\|_{L^\infty}\|a\|_{\dot{B}^\alpha_{p,r}},$$

from this and (2.30), one has

$$\|a(t)\|_{\dot{B}^\alpha_{p,r}} \le \|a_0\|_{\dot{B}^\alpha_{p,r}} + C\int_0^t \|\nabla u\|_{L^\infty}\|a(\tau)\|_{\dot{B}^\alpha_{p,r}}\,d\tau.$$

Gronwall's inequality yields

$$\|a(t)\|_{\dot B^\alpha_{p,r}} \le C\|a_0\|_{\dot B^\alpha_{p,r}} e^{C\|\nabla u\|_{L^1_t(L^\infty)}}, \tag{2.31}$$

from which, we use (2.30) again to deduce that

$$2^{q\alpha}\|\dot\Delta_q a(t)\|_{L^\infty_t(L^p)} \le 2^{q\alpha}\|\dot\Delta_q a_0\|_{L^p} + C\|a_0\|_{\dot B^\alpha_{p,r}}\int_0^t c_{q,r}(\tau)\|\nabla u(\tau)\|_{L^\infty} e^{C\|\nabla u\|_{L^1_\tau(L^\infty)}}\,d\tau.$$

Then, taking the ℓ^r norm in terms of $q \in \{q \ge k\}$ leads to

$$\|a - \dot S_k a\|_{\widetilde L^\infty_t(\dot B^\alpha_{p,r})} \le \Big(\sum_{q\ge k} 2^{rq\alpha}\|\dot\Delta_q a_0\|_{L^p}^r\Big)^{\frac1r} + \|a_0\|_{\dot B^\alpha_{p,r}}\int_0^t C\,U'(\tau)e^{C\,U(\tau)}\,d\tau,$$

which implies (2.28). For $\alpha = 1$, thanks to (2.12), we have

$$\|[u\cdot\nabla, \dot\Delta_q]a\|_{L^2} \lesssim c_{q,1}2^{-q}\|\nabla u\|_{\dot B^0_{\infty,1}}\|a\|_{\dot B^1_{2,1}}.$$

Hence,repeat the above argument, we may get (2.29). □

To prove the uniqueness part of Theorem 1.1, we need the following Propositions.

Proposition 2.4. *Let* $u_0 \in B^{-1}_{2,\infty}(\mathbb{R}^2)$ *and* v *be a divergence free vector field satisfying* $v \in L^1_T(B^1_{\infty,1})$. *Let* $f \in \widetilde L^1_T(B^{-1}_{2,\infty})$, *and* $a \in \widetilde L^\infty_T(B^2_{2,1})$ *with* $1 + a \ge c_1 > 0$ *for some positive constant* c_1. *Assume that* $u \in L^\infty_T(B^{-1}_{2,\infty}) \cap \widetilde L^1_T(B^1_{2,\infty})$ *and* $\nabla\Pi \in \widetilde L^1_T(B^{-1}_{2,\infty})$, *which solves*

$$\begin{cases} \partial_t u + v\cdot\nabla u - (1+a)(\Delta u - \nabla\Pi) = f, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2,\\ \operatorname{div} u = 0,\\ u_{|t=0} = u_0. \end{cases} \tag{2.32}$$

Then there holds:

$$\begin{aligned} \|u\|_{L^\infty_T(B^{-1}_{2,\infty})} + \|u\|_{\widetilde L^1_T(B^1_{2,\infty})} \le & Ce^{C\big(T + T\|\nabla a\|^2_{\widetilde L^\infty_T(B^1_{2,1})} + \|v\|_{L^1_T(B^1_{\infty,1})}\big)}\\ & \times\Big\{\|u_0\|_{B^{-1}_{2,\infty}} + \|f\|_{\widetilde L^1_T(B^{-1}_{2,\infty})} + \|a\|_{\widetilde L^\infty_T(\dot B^1_{2,1})}\|\nabla\Pi\|_{\widetilde L^1_T(B^{-1}_{2,\infty})}\Big\}. \end{aligned} \tag{2.33}$$

Proof. Let $\mathbb{P} \overset{\text{def}}{=} I + \nabla(-\Delta)^{-1}\operatorname{div}$ be the Leray projection operator. Applying $\Delta_q\mathbb{P}$ to (2.32), then a standard commutator process gives

$$\begin{aligned} \partial_t\Delta_q u + (v\cdot\nabla)\Delta_q u - \operatorname{div}((1+a)\nabla\Delta_q u) = [v, \Delta_q\mathbb{P}]\cdot\nabla u - \Delta_q\mathbb{P}(\nabla a\cdot\nabla u)\\ + \Delta_q\mathbb{P}(T_{\nabla a}\Pi) - \Delta_q\mathbb{P}T'_{\nabla\Pi}a + \operatorname{div}(\big[\Delta_q\mathbb{P}, a\big]\nabla u) + \Delta_q\mathbb{P}f. \end{aligned} \tag{2.34}$$

Thanks to the fact that $\operatorname{div} u = \operatorname{div} v = 0$ and $1 + a \ge c_1$, we get by taking the L^2 inner product of (2.34) with $\Delta_q u$ that

$$\begin{aligned} &\frac{d}{dt}\|\Delta_q u\|^2_{L^2} - \int_{\mathbb{R}^2}\operatorname{div}((1+a)\nabla\Delta_q u)\Delta_q u dx\\ &\lesssim \|\Delta_q u\|_{L^2}\Big(\|[v, \Delta_q\mathbb{P}]\cdot\nabla u\|_{L^2} + \|\Delta_q\mathbb{P}(\nabla a\cdot\nabla u)\|_{L^2} + 2^q\|\big[a, \Delta_q\mathbb{P}\big]\nabla u\|_{L^2}\\ &\qquad + \|\Delta_q\mathbb{P}(T_{\nabla a}\Pi)\|_{L^2} + \|\Delta_q\mathbb{P}T'_{\nabla\Pi}a\|_{L^2} + \|\Delta_q\mathbb{P}f\|_{L^2}\Big). \end{aligned}$$

We get, by using integration by parts and Lemma A.5 of [11], that

$$-\int_{\mathbb{R}^2}\operatorname{div}((1+a)\nabla\Delta_q u)\Delta_q u dx = \int_{\mathbb{R}^2}(1+a)|\Delta_q\nabla u|^2\,dx \gtrsim \begin{cases} 2^{2q}\|\Delta_q u\|^2_{L^2}, \quad \forall\, q \ge 0,\\ 0, \quad \text{if} \quad q = -1. \end{cases}$$

This leads to

$$
\begin{aligned}
&\|\Delta_q u\|_{L^\infty_t(L^2)} + 2^{2q}\mathbf{1}_{q\geq 0}\|\Delta_q u\|_{L^1_t(L^2)} \lesssim \|\Delta_q u_0\|_{L^2}\\
&+\int_0^t \big(\|[v,\Delta_q\mathbb{P}]\cdot\nabla u\|_{L^2} + \|\Delta_q\mathbb{P}(\nabla a\cdot\nabla u)\|_{L^2} + 2^q\|\big[\Delta_q\mathbb{P},a\big]\nabla u\|_{L^2}\\
&\quad + \|\Delta_q\mathbb{P}(T_{\nabla a}\Pi)\|_{L^2} + \|\Delta_q\mathbb{P}T'_{\nabla\Pi}a\|_{L^2} + \|\Delta_q\mathbb{P}f\|_{L^2}\big)\,dt'.
\end{aligned}\tag{2.35}
$$

Thanks to (2.14) and (2.15) (up to a zero-order multiplier $\mathbb{P}$ in terms of Δ_q), we deduce from $\operatorname{div} v = 0$ that

$$
\sup_{q\geq -1} 2^{-q}\big\|[v,\Delta_q\mathbb{P}]\cdot\nabla u\big\|_{L^1_T(L^2)} \lesssim \int_0^T \|v\|_{B^1_{\infty,1}}\|u\|_{B^{-1}_{2,\infty}}\,dt \tag{2.36}
$$

and

$$
\sup_{q\geq -1}\|\big[\Delta_q\mathbb{P},a\big]\nabla u\|_{L^1_T(L^2)} \lesssim \big(\|\nabla a\|_{L^\infty_T(L^\infty)} + \|a\|_{L^\infty_T(B^1_{\infty,\infty})}\big)\|u\|_{L^1_T(B^0_{2,1})}. \tag{2.37}
$$

On the other hand, it follows from Bony's decomposition that

$$
\begin{aligned}
&\|\Delta_q\mathbb{P}(\nabla a\cdot\nabla u)\|_{L^1_T(L^2)} \lesssim \|\Delta_q\mathbb{P}T_{\nabla a}\nabla u)\|_{L^1_T(L^2)} + \|\Delta_q\mathbb{P}T'_{\nabla u}\nabla a)\|_{L^1_T(L^2)}\\
&\lesssim \sum_{|k-q|\leq 4}\|S_{k-1}\nabla a\|_{L^\infty_T(L^\infty)}\|\Delta_k\nabla u\|_{L^1_T(L^2)} + 2^q\sum_{k\geq q-3}\|S_{k+2}\nabla u\|_{L^1_T(L^2)}\|\Delta_k\nabla a\|_{L^\infty_T(L^2)}.
\end{aligned}\tag{2.38}
$$

Notice that $\|S_{k-1}\nabla a\|_{L^\infty_T(L^\infty)} \lesssim \|\nabla a\|_{L^\infty_T(L^\infty)}$ and

$$
\|S_{k+2}\nabla u\|_{L^1_T(L^2)} \lesssim \sum_{\ell\leq k+1}\|\Delta_\ell\nabla u\|_{L^1_T(L^2)} \lesssim \|\nabla u\|_{\widetilde{L}^1_T(B^{-1}_{2,1})}2^k c_{k,1},
$$

we then obtain from (2.38) that

$$
\begin{aligned}
&\|\Delta_q\mathbb{P}(\nabla a\cdot\nabla u)\|_{L^1_T(L^2)}\\
&\lesssim \|\nabla a\|_{L^\infty_T(L^\infty)}\sum_{|k-q|\leq 4}\|\Delta_k\nabla u\|_{L^1_T(L^2)} + 2^q\|\nabla u\|_{\widetilde{L}^1_T(B^{-1}_{2,1})}\sum_{k\geq q-3}2^k c_{k,1}\|\Delta_k\nabla a\|_{L^\infty_T(L^2)}\\
&\lesssim 2^q\|\nabla u\|_{\widetilde{L}^1_T(B^{-1}_{2,1})}(\|\nabla a\|_{L^\infty_T(L^\infty)}c_{q,1} + \|\nabla a\|_{L^\infty_T(B^1_{2,\infty})}\sum_{k\geq q-3}c_{k,1})\\
&\lesssim 2^q\|\nabla u\|_{\widetilde{L}^1_T(B^{-1}_{2,1})}\|\nabla a\|_{L^\infty_T(B^1_{2,1})}.
\end{aligned}\tag{2.39}
$$

For $\Delta_q\mathbb{P}T'_{\nabla\Pi}a = \Delta_q\mathbb{P}\sum_{k\geq q-3}S_{k+2}\nabla\Pi\Delta_k a$, we first split it into three parts

$$
\begin{aligned}
\Delta_q\mathbb{P}T'_{\nabla\Pi}a =&\mathbf{1}_{-1\leq q\leq 2}\Delta_q\mathbb{P}(S_1\nabla\Pi\Delta_{-1}a) + \mathbf{1}_{-1\leq q\leq 2}\Delta_q\mathbb{P}\sum_{k\geq 0}S_{k+2}\nabla\Pi\Delta_k a\\
&+ \mathbf{1}_{q\geq 3}\Delta_q\mathbb{P}\sum_{k\geq q-3}S_{k+2}\nabla\Pi\Delta_k a.
\end{aligned}
$$

Owing to Lemma 2.1, one can see

$$
\begin{aligned}
&\|\Delta_q\mathbb{P}T'_{\nabla\Pi}a\|_{L^1_T(L^2)} \lesssim \mathbf{1}_{-1\leq q\leq 2}\|S_1\nabla\Pi\Delta_{-1}a\|_{L^1_T(L^2)}\\
&\quad + \mathbf{1}_{-1\leq q\leq 2}2^q\sum_{k\geq 0}\|S_{k+2}\nabla\Pi\Delta_k a\|_{L^1_T(L^1)} + 2^q\mathbf{1}_{q\geq 3}\sum_{k\geq q-3}\|S_{k+2}\nabla\Pi\Delta_k a\|_{L^1_T(L^1)}\\
&\lesssim \mathbf{1}_{-1\leq q\leq 2}\Big(\|S_1\nabla\Pi\|_{L^1_T(L^2)}\|\Delta_{-1}a\|_{L^\infty_T(L^\infty)} + \sum_{k\geq 0}\|S_{k+2}\nabla\Pi\|_{L^1_T(L^2)}\|\Delta_k a\|_{L^\infty_T(L^2)}\Big)\\
&\quad + 2^q\mathbf{1}_{q\geq 3}\sum_{k\geq q-3}\|S_{k+2}\nabla\Pi\|_{L^1_T(L^2)}\|\Delta_k a\|_{L^\infty_T(L^2)}.
\end{aligned}\tag{2.40}
$$

Thanks to Lemma 2.1 again, we deduce that

$$\|S_{k+2}\nabla\Pi\|_{L^1_T(L^2)} \lesssim \sum_{\ell\leq k+1} \|\Delta_\ell \nabla\Pi\|_{L^1_T(L^2)} \lesssim \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} 2^k c_{k,\infty},$$

which along with (2.40) implies

$$\begin{aligned}\|\Delta_q \mathbb{P} T'_{\nabla\Pi} a\|_{L^1_T(L^2)} \lesssim & \mathbf{1}_{-1\leq q\leq 2} \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} \Big(\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\widetilde{L}^\infty_T(\dot{B}^1_{2,1})} \sum_{k\geq 0} c_{k,1}\Big) \\ & + 2^q \mathbf{1}_{q\geq 3} \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} \|a\|_{\widetilde{L}^\infty_T(\dot{B}^1_{2,1})} \sum_{k\geq q-3} c_{k,1} \\ \lesssim & 2^q \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} \|a\|_{\widetilde{L}^\infty_T(\dot{B}^1_{2,1})}.\end{aligned} \tag{2.41}$$

Similarly, for $\Delta_q \mathbb{P} T_{\nabla a}\Pi = \Delta_q \mathbb{P} \sum_{|k-q|\leq 4} S_{k-1}\nabla a \Delta_k \Pi = \Delta_q \mathbb{P} \sum_{|k-q|\leq 4,\, k\geq 0} S_{k-1}\nabla a \Delta_k\Pi$, we directly deduce from $\|S_{k-1}\nabla a\|_{L^\infty_T(L^2)} \lesssim \|\nabla a\|_{L^\infty_T(L^2)}$ that

$$\begin{aligned}&\|\Delta_q \mathbb{P} T_{\nabla a}\Pi\|_{L^1_T(L^2)} \lesssim 2^q \sum_{|k-q|\leq 4,\, k\geq 0} \|S_{k-1}\nabla a \Delta_k \Pi\|_{L^1_T(L^1)} \\ &\lesssim 2^q \sum_{|k-q|\leq 4,\, k\geq 0} \|S_{k-1}\nabla a\|_{L^\infty_T(L^2)} \|\Delta_k\Pi\|_{L^1_T(L^2)} \\ &\lesssim 2^q \|\nabla a\|_{L^\infty_T(L^2)} \sum_{|k-q|\leq 4,\, k\geq 0} 2^{-k} \|\Delta_k \nabla\Pi\|_{L^1_T(L^2)} \lesssim 2^q \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} \|a\|_{\widetilde{L}^\infty_T(\dot{B}^1_{2,1})}.\end{aligned} \tag{2.42}$$

Plugging (2.36)-(2.42) into (2.35), we arrive at

$$\begin{aligned}\|u\|_{L^\infty_T(B^{-1}_{2,\infty})} + \|u\|_{\widetilde{L}^1_T(B^1_{2,\infty})} \lesssim & \|u_0\|_{B^{-1}_{2,\infty}} + \|\Delta_{-1}u\|_{L^1_T(L^2)} + \int_0^T \|u(t)\|_{B^{-1}_{2,\infty}} \|v(t)\|_{B^1_{\infty,1}}\,dt \\ & + \|\nabla a\|_{\widetilde{L}^\infty_T(B^1_{2,1})} \|u\|_{L^1_T(B^0_{2,1})} + \|a\|_{\widetilde{L}^\infty_T(\dot{B}^1_{2,1})} \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} + \|f\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})},\end{aligned}$$

Thanks to (2.5) and the interpolation inequality (2.7), we have

$$\|\Delta_{-1}u\|_{L^1_T(L^2)} \lesssim \int_0^T \|u(t)\|_{B^{-1}_{2,\infty}}\,dt, \quad \|u\|_{L^1_T(B^0_{2,1})} \lesssim \|u\|^{\frac12}_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} \|u\|^{\frac12}_{\widetilde{L}^1_T(B^1_{2,\infty})}.$$

Hence, we obtain

$$\begin{aligned}\|u\|_{L^\infty_T(B^{-1}_{2,\infty})} + \|u\|_{\widetilde{L}^1_T(B^1_{2,\infty})} \lesssim & \|u_0\|_{B^{-1}_{2,\infty}} + \int_0^T \|u(t)\|_{B^{-1}_{2,\infty}} \big(1 + \|v(t)\|_{B^1_{\infty,1}}\big)dt \\ & + \|\nabla a\|_{\widetilde{L}^\infty_T(B^1_{2,1})} \|u\|^{\frac12}_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} \|u\|^{\frac12}_{\widetilde{L}^1_T(B^1_{2,\infty})} + \|a\|_{\widetilde{L}^\infty_T(\dot{B}^1_{2,1})} \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} + \|f\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})}.\end{aligned}$$

Thanks to Young's inequality, we deduce

$$\begin{aligned}\|u\|_{L^\infty_T(B^{-1}_{2,\infty})} + \|u\|_{\widetilde{L}^1_T(B^1_{2,\infty})} \lesssim & \|u_0\|_{B^{-1}_{2,\infty}} + \int_0^T \|u(t)\|_{B^{-1}_{2,\infty}} \big(1 + \|v(t)\|_{B^1_{\infty,1}}\big)dt \\ & + \|\nabla a\|^2_{\widetilde{L}^\infty_T(B^1_{2,1})} \|u\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} + \|a\|_{\widetilde{L}^\infty_T(\dot{B}^1_{2,1})} \|\nabla\Pi\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})} + \|f\|_{\widetilde{L}^1_T(B^{-1}_{2,\infty})},\end{aligned}$$

which along with Gronwall's inequality yields (2.33). This completes the proof of the proposition. □

Proposition 2.5. *Assume $0 < \underline{b} < \bar{b}$. Let $a \in \dot{B}^1_{2,1}(\mathbb{R}^2)$ such that $0 < \underline{b} \leq 1 + a \leq \bar{b}$, and*

$$\|a - \dot{S}_k a\|_{\dot{B}^1_{2,1}} \leq c \tag{2.43}$$

for some sufficiently small positive constant c and some integer $k \in \mathbb{N}$. Let $F \in B^{-1}_{2,\infty}(\mathbb{R}^2)$ and $\nabla\Pi \stackrel{def}{=} \mathcal{H}_b(F) \in B^{-1}_{2,\infty}(\mathbb{R}^2)$ solves

$$\operatorname{div}((1+a)\nabla\Pi) = \operatorname{div} F. \tag{2.44}$$

Then there holds

$$\|\nabla\Pi\|_{B^{-1}_{2,\infty}} \lesssim (1 + 2^k\|a\|_{B^0_{\infty,1}}(1+\|a\|_{B^0_{\infty,1}}))\big(\|F\|_{B^{-2}_{2,\infty}} + \|\operatorname{div} F\|_{B^{-2}_{2,\infty}}\big). \tag{2.45}$$

Proof. We first deduce from (2.43) and $\underline{b} \leq 1 + a$ that

$$1 + \dot S_k a = 1 + a + (\dot S_k a - a) \geq \frac{b}{2}.$$

Motivated by [2, 14], we shall use a duality argument to prove (2.45). For the sake of simplicity, we just prove (2.45) for sufficiently smooth function F. In order to make the following computation rigorous, one has to use a density argument, which we omit here.

For this, we first estimate $\|\nabla\Pi\|_{B^1_{2,1}}$ under the assumption that $F \in B^1_{2,1}(\mathbb{R}^2)$. Indeed, we write (2.44) as

$$\operatorname{div}[(1+\dot S_k a)\nabla\Pi] = \operatorname{div} F + \operatorname{div}[(\dot S_k a - a)\nabla\Pi],$$

applying Δ_q to the above equation gives

$$\operatorname{div}[(1+\dot S_k a)\Delta_q\nabla\Pi] = \operatorname{div}\Delta_q F + \operatorname{div}\Delta_q[(\dot S_k a - a)\nabla\Pi] + \operatorname{div}([\dot S_k a, \Delta_q]\nabla\Pi).$$

Taking the L^2 inner product of this equation with $\Delta_q\Pi$, we obtain from (2.16) that

$$\begin{aligned}
\|\nabla\Pi\|_{B^1_{2,1}} &\lesssim \|(\dot S_k a - a)\nabla\Pi\|_{B^1_{2,1}} + \|F\|_{B^1_{2,1}} + \sum_{q\geq -1} 2^q\big\|[\dot S_k a, \Delta_q]\nabla\Pi\big\|_{L^2}\\
&\lesssim \|\dot S_k a - a\|_{B^1_{2,1}}\|\nabla\Pi\|_{B^1_{2,1}} + \|F\|_{B^1_{2,1}} + (\|\dot S_k a\|_{B^1_{\infty,1}}\|\nabla\Pi\|_{L^2} + \|\dot S_k a\|_{B^{\frac12}_{\infty,1}}\|\nabla\Pi\|_{H^{\frac12}}).
\end{aligned}$$

Applying the classical elliptic estimate to (2.44), we get $\|\nabla\Pi\|_{L^2} \lesssim \|F\|_{L^2}$. Due to Lemma 2.1, we get

$$\|\dot S_k a\|_{B^1_{\infty,1}} \lesssim \sum_{\ell\leq k-1} 2^\ell\|\Delta_\ell a\|_{L^\infty} \lesssim \|a\|_{B^0_{\infty,1}} \sum_{\ell\leq k-1} 2^\ell c_{\ell,1} \lesssim 2^k\|a\|_{B^0_{\infty,1}},$$

and similar inequality $\|\dot S_k a\|_{B^{\frac12}_{\infty,1}} \lesssim 2^{\frac12}\|a\|_{B^0_{\infty,1}}$.

While by the definition of the Besov space, one can see $\|\dot S_k a - a\|_{B^1_{2,1}} \lesssim \|\dot S_k a - a\|_{\dot B^1_{2,1}}$, for k large. This, together with (2.43) and the interpolation inequality $\|\nabla\Pi\|_{H^{\frac12}} \lesssim \|\nabla\Pi\|^{\frac12}_{B^1_{2,1}}\|\nabla\Pi\|^{\frac12}_{L^2} \lesssim \|\nabla\Pi\|^{\frac12}_{B^1_{2,1}}\|F\|^{\frac12}_{L^2}$, leads to

$$\|\nabla\Pi\|_{B^1_{2,1}} \lesssim \|F\|_{B^1_{2,1}} + (2^k\|a\|_{B^0_{\infty,1}}\|F\|_{L^2} + 2^{\frac12 k}\|a\|_{B^0_{\infty,1}}\|\nabla\Pi\|^{\frac12}_{B^1_{2,1}}\|F\|^{\frac12}_{L^2}).$$

Hence, applying Young's inequality yields

$$\|\nabla\Pi\|_{B^1_{2,1}} \lesssim \big(1 + 2^k\|a\|_{B^0_{\infty,1}}(1+\|a\|_{B^0_{\infty,1}})\big)\|F\|_{B^1_{2,1}}. \tag{2.46}$$

Now we use a duality argument (see Corollary 6.2.8 in [7]) to estimate $\|\nabla\Pi\|_{B^{-1}_{2,\infty}}$ in the case when $F \in B^{-1}_{2,\infty}(\mathbb{R}^2)$. Notice that

$$\|\nabla\Pi\|_{B^{-1}_{2,\infty}} = \sup_{\|g\|_{B^1_{2,1}}\leq 1}\big\langle g, \nabla\Pi\big\rangle = \sup_{\|g\|_{B^1_{2,1}}\leq 1}\Big(-\int \Pi \operatorname{div} g\, dx\Big), \tag{2.47}$$

where $\langle g,\ \nabla\Pi\rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R}^2)$. Whereas (2.46) ensures that for any $g\in B^1_{2,1}(\mathbb{R}^2)$

$$\mathrm{div}\,((1+a)\nabla h_g)=\mathrm{div}\,g$$

has a unique solution $\nabla h_g\in B^1_{2,1}(\mathbb{R}^2)$ satisfying

$$\|\nabla h_g\|_{B^1_{2,1}}\lesssim (1+2^k\|a\|_{B^0_{\infty,1}}(1+\|a\|_{B^0_{\infty,1}}))\|g\|_{B^1_{2,1}}, \tag{2.48}$$

which along with (2.47) yields

$$\begin{aligned}\|\nabla\Pi\|_{B^{-1}_{2,\infty}}&=\sup_{\|g\|_{B^1_{2,1}}\le 1}-\big\langle\Pi,\mathrm{div}\,((1+a)\nabla h_g)\big\rangle=\sup_{\|g\|_{B^1_{2,1}}\le 1}\big\langle(1+a)\nabla\Pi,\nabla h_g\big\rangle\\&=\sup_{\|g\|_{B^1_{2,1}}\le 1}-\big\langle h_g,\mathrm{div}\,((1+a)\nabla\Pi)\big\rangle=\sup_{\|g\|_{B^1_{2,1}}\le 1}-\big\langle \mathrm{div}\,F,h_g\big\rangle\\&=\sup_{\|g\|_{B^1_{2,1}}\le 1}-\big\langle \mathrm{div}\,F,\sum_{\ell\ge -1}\Delta_\ell h_g\big\rangle.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\|\nabla\Pi\|_{B^{-1}_{2,\infty}}&=\sup_{\|g\|_{B^1_{2,1}}\le 1}-\big\langle\mathrm{div}\,F,\Delta_{-1}h_g\big\rangle+\sup_{\|g\|_{B^1_{2,1}}\le 1}-\big\langle\mathrm{div}\,F,\sum_{\ell\ge 0}\Delta_\ell h_g\big\rangle\\&=\sup_{\|g\|_{B^1_{2,1}}\le 1}\big\langle F,\nabla\Delta_{-1}h_g\big\rangle+\sup_{\|g\|_{B^1_{2,1}}\le 1}-\big\langle\mathrm{div}\,F,\sum_{\ell\ge 0}\Delta_\ell h_g\big\rangle.\end{aligned}$$

Whence thanks to (2.48), we obtain

$$\begin{aligned}\|\nabla\Pi\|_{B^{-1}_{2,\infty}}&\lesssim\sup_{\|g\|_{B^1_{2,1}}\le 1}\|F\|_{B^{-2}_{2,\infty}}\|\nabla\Delta_{-1}h_g\|_{B^2_{2,1}}+\sup_{\|g\|_{B^1_{2,1}}\le 1}\|\mathrm{div}\,F\|_{B^{-2}_{2,\infty}}\|\sum_{\ell\ge 0}\Delta_\ell h_g\|_{B^2_{2,1}}\\&\lesssim\sup_{\|g\|_{B^1_{2,1}}\le 1}\|F\|_{B^{-2}_{2,\infty}}\|\nabla h_g\|_{B^1_{2,1}}+\sup_{\|g\|_{B^1_{2,1}}\le 1}\|\mathrm{div}\,F\|_{B^{-2}_{2,\infty}}\|\nabla h_g\|_{B^1_{2,1}}\\&\lesssim(1+2^k\|a\|_{B^0_{\infty,1}}(1+\|a\|_{B^0_{\infty,1}}))\big(\|F\|_{B^{-2}_{2,\infty}}+\|\mathrm{div}\,F\|_{B^{-2}_{2,\infty}}\big),\end{aligned}$$

which completes the proof of this proposition. $\square$

In order to get the uniqueness of the solution in the critical case in Theorem 1.1, we need to recall the following Osgood's lemma [18].

Lemma 2.4 ([18], Osgood's lemma). *Let $f\ge 0$ be a measurable function, γ be a locally integrable function and μ be a positive, continuous and nondecreasing function which verifies the following condition*

$$\int_0^1\frac{dr}{\mu(r)}=+\infty.$$

Let also a be a positive real number and let f satisfy the inequality

$$f(t)\le a+\int_0^t\gamma(s)\mu(f(s))\,ds.$$

Then if a is equal to zero, the function f vanishes. If a is not zero, then we have

$$-\mathcal{M}(f(t))+\mathcal{M}(a)\le\int_0^t\gamma(s)\,ds\quad\text{with}\quad\mathcal{M}(x)=\int_x^1\frac{dr}{\mu(r)}.$$

3. The $L^1([0,T];\dot B^2_{2,1})$ estimate for the velocity field

In this section, we want to get, at least in the small time interval, the $L^1([0,T];\dot B^2_{2,1})$ estimate for the velocity field, which plays a crucial role in the study of the uniqueness of the solution to (1.1). For this, we first investigate some *a priori* estimates about the basic energy and the pressure.

Proposition 3.1. *Let $\varepsilon\in(0,1)$, $a_0:=\rho_0^{-1}-1\in \dot B^0_{\infty,1}\cap \dot B^{\varepsilon}_{\frac{2}{\varepsilon},\infty}(\mathbb{R}^2)$ and $u_0\in L^2(\mathbb{R}^2)$, and (1.5) holds. Let $(\rho,u,\nabla\Pi)$ be a smooth enough solution of (1.1) on $[0,T^*[$, then for any $t\in]0,T^*[$, there hold that*

$$\|\sqrt{\rho}u\|^2_{L^\infty_t(L^2)}+2\|\nabla u\|^2_{L^2_t(L^2)}\leq\|\sqrt{\rho_0}u_0\|^2_{L^2}, \tag{3.1}$$

and

$$\begin{aligned}\|\nabla\Pi\|_{L^1_t(L^2)}\leq&\Big(\eta+M\sum_{q\geq k}\|\dot\Delta_q a_0\|_{L^\infty}+M\|a_0\|_{\dot B^0_{\infty,1}}(e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1)\Big)\|\Delta u\|_{L^1_t(L^2)}\\&+C_\eta\Big(\sqrt{t}2^k+\sqrt{t}2^ke^{C\|\nabla u\|_{L^1_t(L^\infty)}}+\|\nabla u\|^2_{L^2_t(L^2)}\Big)\end{aligned} \tag{3.2}$$

for any positive constant η.

Proof. We first get, by using standard energy estimate to (1.2), that

$$\frac12\frac{d}{dt}\|\sqrt{\rho}u\|^2_{L^2}+\|\nabla u\|^2_{L^2}=0,$$

which immediately follows (3.1).

On the other hand, let $a\stackrel{\text{def}}{=}\rho^{-1}-1$, the system (1.2) can be equivalently reformulated as

$$\begin{cases}\partial_t a+u\cdot\nabla a=0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^2,\\ \partial_t u+(u\cdot\nabla)u-(1+a)(\Delta u-\nabla\Pi)=0,\\ \operatorname{div} u=0,\\ (a,u_0)|_{t=0}=(a_0,u_0).\end{cases} \tag{3.3}$$

Applying the maximum principle and Lemma 2.3 to the first equation in (3.3) gives rise to

$$\|a(t)\|_{L^\infty}\leq\|a_0\|_{L^\infty}\quad\text{and}\quad \|a\|_{L^\infty_t(\dot B^{\varepsilon}_{\frac{2}{\varepsilon},\infty})}\leq\|a_0\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},\infty}}e^{C\|\nabla u\|_{L^1_t(L^\infty)}}. \tag{3.4}$$

Applying the div operator to the momentum equation of (3.3) yields that

$$\operatorname{div}\big((1+a)\nabla\Pi\big)=\operatorname{div}(a\Delta u)-\operatorname{div}\big((u\cdot\nabla)u\big), \tag{3.5}$$

for some large enough integer k we shall rewrite the above equality as

$$\begin{aligned}\operatorname{div}\big((1+a)\nabla\Pi\big)&=\operatorname{div}\big((a-\dot S_k a)\Delta u\big)+\operatorname{div}(\dot S_k a\Delta u)-\operatorname{div}\big((u\cdot\nabla)u\big)\\&=\operatorname{div}\big((a-\dot S_k a)\Delta u\big)+T_{\Delta u}\nabla\dot S_k a+T_{\nabla\dot S_k a}\Delta u+\operatorname{div}\big(R(\dot S_k a,\Delta u)\big)-\operatorname{div}\big((u\cdot\nabla)u\big).\end{aligned}$$

By taking L^2 inner product of the above equation with Π, we get from the fact $1+a=\frac{1}{\rho}\geq\frac{1}{M}$ that

$$\begin{aligned}\frac{1}{M}\|\nabla\Pi\|^2_{L^2}\leq\|\nabla\Pi\|_{L^2}\Big(&\|(a-\dot S_k a)\Delta u\|_{L^2}+\|T_{\Delta u}\nabla\dot S_k a\|_{\dot H^{-1}}+\|T_{\nabla\dot S_k a}\Delta u\|_{\dot H^{-1}}\\&+\|R(\dot S_k a,\Delta u)\|_{L^2}+\|(u\cdot\nabla)u\|_{L^2}\Big).\end{aligned} \tag{3.6}$$

Thanks to product laws in Besov spaces Proposition (2.2), one can see

$$\begin{aligned}
&\|(a-\dot S_k a)\Delta u\|_{L^2}\le \|a-\dot S_k a\|_{L^\infty}\|\Delta u\|_{L^2},\\
&\|T_{\Delta u}\nabla \dot S_k a\|_{\dot H^{-1}}+\|T_{\nabla \dot S_k a}\Delta u\|_{\dot H^{-1}}\lesssim \|\nabla \dot S_k a\|_{L^\infty}\|\Delta u\|_{\dot H^{-1}}\lesssim \|\nabla \dot S_k a\|_{L^\infty}\|\nabla u\|_{L^2},\\
&\|R(\dot S_k a,\Delta u)\|_{L^2}\lesssim \|R(\dot S_k a,\Delta u)\|_{\dot B^{\varepsilon}_{\frac{2}{1+\varepsilon},2}}\lesssim \|\dot S_k a\|_{\dot B^{\varepsilon+1}_{\frac{2}{\varepsilon},\infty}}\|\nabla u\|_{L^2}\lesssim 2^k\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},\infty}}\|\nabla u\|_{L^2},\\
&\|(u\cdot\nabla)u\|_{L^2}\lesssim \|u\|_{L^4}\|\nabla u\|_{L^4}\lesssim \|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}.
\end{aligned}$$

from which and (3.6), we infer

$$\begin{aligned}
\|\nabla\Pi\|_{L^2}\le& M\|a-\dot S_k a\|_{L^\infty}\|\Delta u\|_{L^2}+C\,2^k\|a\|_{L^\infty}\|\nabla u\|_{L^2}\\
&+C\,2^k\|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},\infty}}\|\nabla u\|_{L^2}+C\,\|u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}.
\end{aligned}\tag{3.7}$$

Thus by Young's inequality, we deduce from (3.4) that for any positive constant η

$$\begin{aligned}
\|\nabla\Pi\|_{L^1_t(L^2)}\le&\big(M\|a-\dot S_k a\|_{L^\infty_t(L^\infty)}+\eta\big)\|\Delta u\|_{L^1_t(L^2)}+C_\eta\sqrt{t}2^k\|a_0\|_{L^\infty}\|\nabla u\|_{L^2_t(L^2)}\\
&+C_\eta\sqrt{t}2^k\|a_0\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},\infty}}\|\nabla u\|_{L^2_t(L^2)}e^{C\|\nabla u\|_{L^1_t(L^\infty)}}+C_\eta\,\|u\|_{L^\infty_t(L^2)}\|\nabla u\|_{L^2_t(L^2)}^2.
\end{aligned}\tag{3.8}$$

Thanks to Proposition 2.3, we have

$$\|a-\dot S_k a\|_{L^\infty_t(L^\infty)}\le \|a-\dot S_k a\|_{\widetilde L^\infty_t(\dot B^0_{\infty,1})}\le \sum_{q\ge k}\|\dot\Delta_q a_0\|_{L^\infty}+\|a_0\|_{\dot B^0_{\infty,1}}(e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1).\tag{3.9}$$

Plugging (3.9) into (3.8) ensures (3.2). This completes the proof of Proposition 3.1. $\square$

With Proposition 3.1 in hand, we are in a position to prove the following proposition about the estimate $\|u\|_{L^1_t(\dot B^2_{2,1})}$.

Proposition 3.2. *Let $\varepsilon\in(0,1)$, $u_0\in \dot B^0_{2,1}$ and $a_0\in \dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}$. Let $(a,u,\nabla\Pi)$ be a smooth enough solution of (3.3) on $[0,T^*[$, then there is small positive time $T_1<T^*$ such that, for all $t\le T_1$, there holds*

$$\|u\|_{L^1_t(\dot B^2_{2,1})}+\|\nabla u\|_{L^2_t(L^2)}\lesssim \sum_{j\in\mathbb Z}\big(1-e^{-ct2^{2j}}\big)\|\dot\Delta_j u_0\|_{L^2}+\sqrt{t}.\tag{3.10}$$

Proof. Let $\mathbb P\stackrel{\text{def}}{=}I+\nabla(-\Delta)^{-1}\mathrm{div}$ be the Leray projection operator. We get, by first dividing the momentum equation of (1.2) by ρ and then applying the resulting equation by the operator $\mathbb P$, that

$$\partial_t u+\mathbb P\big(u\cdot\nabla u\big)-\mathbb P\big(\rho^{-1}(\Delta u-\nabla\Pi)\big)=0.$$

Applying $\dot\Delta_j$ to the above equation and using a standard commutator's process, we write

$$\rho\partial_t\dot\Delta_j u+\rho u\cdot\nabla\dot\Delta_j u-\Delta\dot\Delta_j u=-\rho[\dot\Delta_j\mathbb P,u\cdot\nabla]u+\rho[\dot\Delta_j\mathbb P,\rho^{-1}]\big(\Delta u-\nabla\Pi\big).\tag{3.11}$$

Taking L^2 inner product of (3.11) with $\dot\Delta_j u$, we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_{\mathbb R^2}\rho|\dot\Delta_j u|^2\,dx-\int_{\mathbb R^2}\Delta\dot\Delta_j u\cdot\dot\Delta_j u\,dx\\
&\le \|\dot\Delta_j u\|_{L^2}\Big(\|\rho[\Delta_j\mathbb P,u\cdot\nabla]u\|_{L^2}+\|\rho[\dot\Delta_j\mathbb P,\rho^{-1}]\big(\Delta u-\nabla\Pi\big)\|_{L^2}\Big).
\end{aligned}\tag{3.12}$$

We get, by using integration by parts and Lemma 2.1, that

$$-\int_{\mathbb R^2}\Delta\dot\Delta_j u\cdot\Delta_j u\,dx=\int_{\mathbb R^2}|\nabla\dot\Delta_j u|^2\,dx\ge \bar c2^{2j}\|\dot\Delta_j u\|_{L^2}^2.$$

Thus, we deduce from (3.12) that

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}\dot{\Delta}_j u\|_{L^2}^2+2c2^{2j}\|\sqrt{\rho}\dot{\Delta}_j u\|_{L^2}^2\\&\lesssim\|\sqrt{\rho}\dot{\Delta}_j u\|_{L^2}(\|[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u\|_{L^2}+\|[\dot{\Delta}_j\mathbb{P},\rho^{-1}]\big(\Delta u-\nabla\Pi\big)\|_{L^2}),\end{aligned}$$

where $c=\bar{c}/M$. This gives rise to

$$\begin{aligned}&\|\sqrt{\rho}\dot{\Delta}_j u(t)\|_{L^2}\lesssim e^{-c2^{2j}t}\|\sqrt{\rho_0}\dot{\Delta}_j u_0\|_{L^2}\\&+\int_0^t e^{-c2^{2j}(t-t')}\Big(\|[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u\|_{L^2}+\|[\dot{\Delta}_j\mathbb{P},\rho^{-1}]\big(\Delta u-\nabla\Pi\big)\|_{L^2}\Big)(t')\,dt'.\end{aligned}\tag{3.13}$$

As a consequence, by virtue of Definition 2.2, we infer

$$\begin{aligned}\|u\|_{L^1_t(\dot{B}^2_{2,1})}\lesssim&\sum_{j\in\mathbb{Z}}\big(1-e^{-ct2^{2j}}\big)\|\dot{\Delta}_j u_0\|_{L^2}+\sum_{j\in\mathbb{Z}}\|[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u\|_{L^1_t(L^2)}\\&+\sum_{j\in\mathbb{Z}}\|[\dot{\Delta}_j\mathbb{P},\rho^{-1}]\big(\Delta u-\nabla\Pi\big)\|_{L^1_tL^2}.\end{aligned}\tag{3.14}$$

In what follows, we shall deal with term by term the right-hand side of (3.14). Thanks to (2.13), we first obtain

$$\sum_{j\in\mathbb{Z}}\|[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u\|_{L^1_t(L^2)}\lesssim\|\nabla u\|_{L^2_t(L^2)}\|u\|_{L^2_t(\dot{B}^1_{2,1})}.$$

from this, we use the interpolation inequality (2.6), $\|u\|_{L^2_t(\dot{B}^1_{2,1})}\lesssim\|u\|^{\frac12}_{L^\infty_t(L^2)}\|u\|^{\frac12}_{L^1_t(\dot{B}^2_{2,1})}$, and Young's inequality to find

$$\sum_{j\in\mathbb{Z}}\|[\dot{\Delta}_j\mathbb{P},u\cdot\nabla]u\|_{L^1_t(L^2)}\leq C_\eta\|u\|_{L^\infty_t(L^2)}\|\nabla u\|^2_{L^2_t(L^2)}+\eta\|u\|_{L^1_t(\dot{B}^2_{2,1})}\tag{3.15}$$

for any positive constant η.

While thanks to (2.17), we deduce that

$$\sum_{j\in\mathbb{Z}}\|[\dot{\Delta}_j\mathbb{P},\rho^{-1}]\nabla\Pi\|_{L^1_tL^2}\lesssim\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})}\|\nabla\Pi\|_{L^1_t(L^2)}.\tag{3.16}$$

On the other hand, note that

$$[\dot{\Delta}_j\mathbb{P},\rho^{-1}]\Delta u=[\dot{\Delta}_j\mathbb{P},a]\Delta u=[\dot{\Delta}_j\mathbb{P},a-\dot{S}_ka]\Delta u+[\dot{\Delta}_j\mathbb{P},\dot{S}_ka]\Delta u.\tag{3.17}$$

Using (2.17) again, we obtain

$$\sum_{j\in\mathbb{Z}}\|[\dot{\Delta}_j\mathbb{P},a-\dot{S}_ka]\Delta u\|_{L^1_tL^2}\lesssim\|a-\dot{S}_ka\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})}\|\Delta u\|_{L^1_t(L^2)}.\tag{3.18}$$

For $[\dot{\Delta}_j\mathbb{P},\dot{S}_ka]\Delta u$, we deduce from (2.18) that

$$\sum_{j\in\mathbb{Z}}\big\|[\dot{\Delta}_j\mathbb{P},\dot{S}_ka]\Delta u\big\|_{L^1_t(L^2)}\lesssim\|\nabla\dot{S}_ka\|_{L^\infty_t(L^\infty)}\|u\|_{L^1_t(\dot{B}^1_{2,1})}+\|\dot{S}_ka\|_{L^\infty_t(\dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1})}\|\nabla u\|_{L^1_t(L^2)}$$

which along with the interpolation inequality (2.6), $\|u\|_{\dot{B}^1_{2,1}} \lesssim \|u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}$, and Young's inequality leads to

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\big\|[\dot{\Delta}_j\mathbb{P}, \dot{S}_k a]\Delta u\big\|_{L^1_t(L^2)}\\
&\leq C(\|\nabla \dot{S}_k a\|_{L^\infty_t(L^\infty)}\sqrt{t}\|u\|^{\frac{1}{2}}_{L^\infty_t(L^2)}\|\Delta u\|^{\frac{1}{2}}_{L^1_t(L^2)} + \|\dot{S}_k a\|_{L^\infty_t(\dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1})}\sqrt{t}\|\nabla u\|_{L^2_t(L^2)})\\
&\leq \gamma\|\Delta u\|_{L^1_t(L^2)} + C_\gamma\, t\,\|\nabla \dot{S}_k a\|^2_{L^\infty_t(L^\infty)}\|u\|_{L^\infty_t(L^2)} + C\sqrt{t}\,\|\dot{S}_k a\|_{L^\infty_t(\dot{B}^{1+\varepsilon}_{\frac{2}{\varepsilon},1})}\|\nabla u\|_{L^2_t(L^2)}
\end{aligned}$$

for any positive constant γ.

Hence, due to Lemma 2.1, one has

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\big\|[\dot{\Delta}_j\mathbb{P}, \dot{S}_k a]\Delta u\big\|_{L^1_t(L^2)} \leq& \gamma\|\Delta u\|_{L^1_t(L^2)} + C_\gamma\, t\, 2^{2k}\|a\|^2_{L^\infty_t(L^\infty)}\|u\|_{L^\infty_t(L^2)}\\
&+ C\sqrt{t}\,2^k\|a\|_{L^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})}\|\nabla u\|_{L^2_t(L^2)}.
\end{aligned}\tag{3.19}$$

Combining (3.18), (3.19) with (3.17) gives rise to

$$\begin{aligned}
\sum_{j\in\mathbb{Z}}\big\|[\dot{\Delta}_j\mathbb{P}, \rho^{-1}]\Delta u\big\|_{L^1_t(L^2)} \leq& (\gamma + C\|a-\dot{S}_k a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})})\|\Delta u\|_{L^1_t(L^2)}\\
&+ C_\gamma t 2^{2k}\|a\|^2_{L^\infty}\|u\|_{L^\infty_t(L^2)} + C\sqrt{t}2^k\|a\|_{L^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})}\|\nabla u\|_{L^2_t(L^2)}.
\end{aligned}\tag{3.20}$$

Substituting (3.15), (3.16) and (3.20) into (3.14), and taking η and γ small enough, we obtain

$$\begin{aligned}
\|u\|_{L^1_t(\dot{B}^2_{2,1})} \leq& C\sum_{j\in\mathbb{Z}}\big(1-e^{-ct2^{2j}}\big)\|\dot{\Delta}_j u_0\|_{L^2} + C\|\nabla u\|^2_{L^2_t(L^2)}\\
&+ C\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})}\|\nabla\Pi\|_{L^1_t(L^2)} + C\|a-\dot{S}_k a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})}\|\Delta u\|_{L^1_t(L^2)}\\
&+ C\sqrt{t}2^k(\sqrt{t} + \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})}).
\end{aligned}\tag{3.21}$$

Thanks to (2.31) and (2.28), we get

$$\begin{aligned}
&\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})} + \|a\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})} \lesssim e^{C\|\nabla u\|_{L^1_t(L^\infty)}},\\
&\|a-\dot{S}_k a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})} \leq \sum_{q\geq k}2^{q\varepsilon}\|\dot{\Delta}_q a_0\|_{L^{\frac{2}{\varepsilon}}} + \|a_0\|_{\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}}(e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1).
\end{aligned}\tag{3.22}$$

Therefore, thanks to (3.2) and (3.22), we obtain from (3.21) that

$$\begin{aligned}
\|u\|_{L^1_t(\dot{B}^2_{2,1})} \leq& C\sum_{j\in\mathbb{Z}}\big(1-e^{-ct2^{2j}}\big)\|\dot{\Delta}_j u_0\|_{L^2} + Ce^{C\|\nabla u\|_{L^1_t(L^\infty)}}\Big(\sqrt{t}2^k + \|\nabla u\|^2_{L^2_t(L^2)}\Big)\\
&+ Ce^{C\|\nabla u\|_{L^1_t(L^\infty)}}\Big(\eta + M\sum_{q\geq k}\|\dot{\Delta}_q a_0\|_{L^\infty} + M\|a_0\|_{\dot{B}^0_{\infty,1}}\{e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1\}\Big)\|u\|_{L^1_t(\dot{B}^2_{2,1})}\\
&+ C\Big(\sum_{q\geq k}2^{q\varepsilon}\|\dot{\Delta}_q a_0\|_{L^{\frac{2}{\varepsilon}}} + \|a_0\|_{\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}}(e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1)\Big)\|u\|_{L^1_t(\dot{B}^2_{2,1})}.
\end{aligned}\tag{3.23}$$

By using (3.13) again, we deduce from the fact $\ell^1 \hookrightarrow \ell^2$ that

$$\|\nabla u\|_{L^2_t(L^2)} \lesssim \Big(\sum_{j\in\mathbb{Z}}\big(1-e^{-ct2^{2j}}\big)\|\dot\Delta_j u_0\|_{L^2}^2\Big)^{\frac12} + \Big(\sum_{j\in\mathbb{Z}}\|[\dot\Delta_j\mathbb{P}, u\cdot\nabla]u\|^2_{L^1_t(L^2)}\Big)^{\frac12} + \sum_{j\in\mathbb{Z}}\|[\dot\Delta_j\mathbb{P},\rho^{-1}]\big(\Delta u-\nabla\Pi\big)\|_{L^1_tL^2}.$$

Thanks to (2.13), we get

$$\Big(\sum_{j\in\mathbb{Z}}\|[\dot\Delta_j\mathbb{P}, u\cdot\nabla]u\|^2_{L^1_t(L^2)}\Big)^{\frac12} \lesssim \|\nabla u\|^2_{L^2_t(L^2)}.$$

Hence, from (3.16), (3.20), and (3.2), it follows that

$$\begin{aligned}\|\nabla u\|_{L^2_t(L^2)} &\lesssim \Big(\sum_{j\in\mathbb{Z}}\big(1-e^{-ct2^{2j}}\big)\|\dot\Delta_j u_0\|_{L^2}^2\Big)^{\frac12} + Ce^{C\|\nabla u\|_{L^1_t(L^\infty)}}\Big(\sqrt{t}2^k + \|\nabla u\|^2_{L^2_t(L^2)}\Big)\\ &+ Ce^{C\|\nabla u\|_{L^1_t(L^\infty)}}\Big(\eta + M\sum_{q\ge k}\|\dot\Delta_q a_0\|_{L^\infty} + M\|a_0\|_{\dot B^0_{\infty,1}}\{e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1\}\Big)\|u\|_{L^1_t(\dot B^2_{2,1})}\\ &+ C\Big(\sum_{q\ge k}2^{q\varepsilon}\|\dot\Delta_q a_0\|_{L^{\frac{2}{\varepsilon}}} + \|a_0\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}(e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1)\Big)\|u\|_{L^1_t(\dot B^2_{2,1})}.\end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24), we achieve

$$\begin{aligned}&\|u\|_{L^1_t(\dot B^2_{2,1})} + \|\nabla u\|_{L^2_t(L^2)}\\ &\le C\sum_{j\in\mathbb{Z}}\big(1-e^{-ct2^{2j}}\big)\|\dot\Delta_j u_0\|_{L^2} + Ce^{C\|\nabla u\|_{L^1_t(L^\infty)}}\Big(\sqrt{t}2^k + \|\nabla u\|^2_{L^2_t(L^2)}\Big)\\ &+ Ce^{C\|\nabla u\|_{L^1_t(L^\infty)}}\Big(\eta + M\sum_{q\ge k}\|\dot\Delta_q a_0\|_{L^\infty} + M\|a_0\|_{\dot B^0_{\infty,1}}\{e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1\}\Big)\|u\|_{L^1_t(\dot B^2_{2,1})}\\ &+ C\Big(\sum_{q\ge k}2^{q\varepsilon}\|\dot\Delta_q a_0\|_{L^{\frac{2}{\varepsilon}}} + \|a_0\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}(e^{C\|\nabla u\|_{L^1_t(L^\infty)}}-1)\Big)\|u\|_{L^1_t(\dot B^2_{2,1})}.\end{aligned} \tag{3.25}$$

Consequently, taking η small enough, k large enough, and then t sufficiently small in (3.25), we deduce (3.10), which completes the proof of of Proposition 3.2. $\square$

Based on this, we may get further estimate about the pressure.

Proposition 3.3. *Let $\varepsilon\in(0,1)$ and $a\in\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}(\mathbb{R}^2)$ such that $0<\underline{b}\le 1+a\le\bar b$, and*

$$\|a-\dot S_k a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}} \le c \tag{3.26}$$

for some sufficiently small positive constant c and some integer $k\in\mathbb{Z}$. Let $F\in\dot B^0_{2,1}(\mathbb{R}^2)$ and $\nabla\Pi \overset{def}{=} \mathcal{H}_b(F)\in\dot B^0_{2,1}(\mathbb{R}^2)$ solves

$$\operatorname{div}\big((1+a)\nabla\Pi\big) = \operatorname{div} F. \tag{3.27}$$

Then there holds

$$\|\nabla\Pi\|_{\dot B^0_{2,1}} \lesssim \|F\|_{\dot B^0_{2,1}} + \|a\|_{\dot B^{\varepsilon}_{\frac{2}{\varepsilon},1}}\|\nabla\Pi\|_{L^2}. \tag{3.28}$$

Proof. We first deduce from (3.26) and $\underline{b} \leq 1 + a$ that

$$1 + \dot{S}_k a = 1 + a + (\dot{S}_k a - a) \geq \frac{\underline{b}}{2}.$$

We rewrite (3.27) in the following form

$$\mathrm{div}[(1 + \dot{S}_k a)\nabla\Pi] = \mathrm{div}\, F + \mathrm{div}\, [(\dot{S}_k a - a)\nabla\Pi],$$

and applying $\dot{\Delta}_q$ to the above equation gives

$$\mathrm{div}\, [(1 + \dot{S}_k a)\dot{\Delta}_q\nabla\Pi] = \mathrm{div}\, \dot{\Delta}_q F + \mathrm{div}\, \dot{\Delta}_q[(\dot{S}_k a - a)\nabla\Pi] + \mathrm{div}\, ([\dot{S}_k a, \dot{\Delta}_q]\nabla\Pi).$$

Taking the L^2 inner product of this equation with $\dot{\Delta}_q\Pi$, we obtain that

$$\|\nabla\Pi\|_{\dot{B}^0_{2,1}} \lesssim \|(\dot{S}_k a - a)\nabla\Pi\|_{\dot{B}^0_{2,1}} + \|F\|_{\dot{B}^0_{2,1}} + \sum_{q\in\mathbb{Z}} \big\|[\dot{S}_k a, \dot{\Delta}_q]\nabla\Pi\big\|_{L^2},$$

which follows from the product law in Besov spaces and (2.17) that

$$\|\nabla\Pi\|_{\dot{B}^0_{2,1}} \lesssim \|\dot{S}_k a - a\|_{\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}} \|\nabla\Pi\|_{\dot{B}^0_{2,1}} + \|F\|_{\dot{B}^0_{2,1}} + \|a\|_{\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}} \|\nabla\Pi\|_{L^2},$$

Hence, due to (3.26), we deduce (3.28), which completes the proof of the proposition. $\square$

Consequently, we may derive the main result of the section as follows.

Proposition 3.4. *Under the assumptions of Proposition 3.2, there holds that for any* $t \in [0, T_1]$

$$\|u\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})} + \|u\|_{L^1_t(\dot{B}^2_{2,1})} + \|\partial_t u\|_{L^1_t(\dot{B}^0_{2,1})} + \|\nabla\Pi\|_{L^1_t(\dot{B}^0_{2,1})} \leq C_0, \tag{3.29}$$

where the constant C_0 *depends only on the initial data* (ρ_0, u_0), *and the positive time* T_1 *is determined by Proposition 3.2.*

Proof. Back to the proof of Proposition 3.2, according to (3.5), we have

$$\mathrm{div}\big((1 + a)\nabla\Pi\big) = \mathrm{div}(a\Delta u) - \mathrm{div}\big((u \cdot \nabla)u\big). \tag{3.30}$$

Combining Proposition 2.3 with Proposition 3.2, we know that the inequality (3.26) holds for any $t \in [0, T_1]$. Then applying Proposition 2.5 to (3.30) yields that

$$\|\nabla\Pi\|_{\dot{B}^0_{2,1}} \lesssim \|a\Delta u\|_{\dot{B}^0_{2,1}} + \|(u \cdot \nabla)u\|_{\dot{B}^0_{2,1}} + \|a\|_{\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}} \|\nabla\Pi\|_{L^2},$$

and then

$$\|\nabla\Pi\|_{L^1_t(\dot{B}^0_{2,1})} \lesssim \|a\Delta u\|_{L^1_t(\dot{B}^0_{2,1})} + \|(u \cdot \nabla)u\|_{L^1_t(\dot{B}^0_{2,1})} + \|a\|_{L^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})} \|\nabla\Pi\|_{L^1_t(L^2)}. \tag{3.31}$$

Due to product laws in Besov spaces (Proposition 2.2) and the interpolation inequality (2.6), we get $\|a\Delta u\|_{L^1_t(\dot{B}^0_{2,1})} \lesssim \|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})} \|u\|_{L^1_t(\dot{B}^2_{2,1})}$ and

$$\begin{aligned}\|(u \cdot \nabla)u\|_{L^1_t(\dot{B}^0_{2,1})} &\lesssim \|\mathrm{div}\,(u \otimes u)\|_{L^1_t(\dot{B}^0_{2,1})} \lesssim \|u \otimes u\|_{L^1_t(\dot{B}^1_{2,1})} \lesssim \int_0^t \|u\|^2_{\dot{B}^1_{2,1}}\, d\tau \\ &\lesssim \int_0^t \|u\|_{L^2} \|u\|_{\dot{B}^2_{2,1}}\, d\tau \lesssim \|u\|_{L^\infty_t(L^2)} \|u\|_{L^1_t(\dot{B}^2_{2,1})},\end{aligned}$$

Hence, thanks to Propositions 3.1, 3.2, and (2.31), we achieve

$$\begin{aligned}&\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})} \leq C_0, \quad \|\nabla\Pi\|_{L^1_t(L^2)} \leq C_0, \quad \|u\|_{L^1_t(\dot{B}^2_{2,1})} \leq C_0, \\ &\|a\Delta u\|_{L^1_t(\dot{B}^0_{2,1})} \leq C_0, \quad \|(u \cdot \nabla)u\|_{L^1_t(\dot{B}^0_{2,1})} \leq C_0.\end{aligned} \tag{3.32}$$

Inserting (3.32) into (3.31) ensures that

$$\|\nabla\Pi\|_{L^1_t(\dot{B}^0_{2,1})} \leq C_0.$$

On the other hand, thanks to (3.13), (3.15), (3.16) and (3.20), we readily deduce that

$$\|u\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})} \le C_0.$$

While from the momentum equations in (1.2) and (3.32), one has

$$\begin{aligned}\|\partial_t u\|_{L^1_t(\dot{B}^0_{2,1})} &\lesssim \|(u\cdot\nabla)u\|_{L^1_t(\dot{B}^0_{2,1})} + \|\Delta u-\nabla\Pi\|_{L^1_t(\dot{B}^0_{2,1})} + \|a(\Delta u-\nabla\Pi)\|_{L^1_t(\dot{B}^0_{2,1})}\\ &\le C_0 + C(1+\|a\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})})(\|u\|_{L^1_t(\dot{B}^2_{2,1})} + \|\nabla\Pi\|_{L^1_t(\dot{B}^0_{2,1})}) \le C_0,\end{aligned}$$

which follows (3.29). This ends the proof of Proposition 3.4. □

4. The proof of Theorem 1.1

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into two steps.

Step 1. Existence of strong solutions.

Given ρ_0 with $a_0 := \rho_0^{-1} - 1 \in \dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}(\mathbb{R}^2)$ and satisfying (1.5), $u_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)$, we first mollify the initial data to be

$$a_{0,n} \stackrel{\text{def}}{=} a_0 * j_n, \quad \text{and} \quad u_{0,n} \stackrel{\text{def}}{=} u_0 * j_n, \tag{4.1}$$

where $j_n(|x|) = n^2 j(|x|/n)$ is the standard Friedrich's mollifier. Then we deduce from the standard well-posedness theory of inhomogeneous Navier-Stokes system (see [13] for instance) that (1.2) has a unique solution $(\rho_n, u_n, \nabla\Pi_n)$ on $[0, T_n^*[$ for some positive time T_n^*. It is easy to observe from (4.1) that

$$\|a_{0,n}\|_{\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}} \le C\|a_0\|_{\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}} \quad \text{and} \quad \|u_{0,n}\|_{\dot{B}^0_{2,1}} \le C\|u_0\|_{B^0_{2,1}},$$

so, under the assumptions of Theorem 1.1, we infer from Propositions 3.2 and 3.4 that there holds

$$\begin{aligned}&\|u_n\|_{\widetilde{L}^\infty_t(\dot{B}^0_{2,1})} + \|u_n\|_{L^1_t(\dot{B}^2_{2,1})} + \|\partial_t u_n\|_{L^1_t(\dot{B}^0_{2,1})} + \|\nabla\Pi_n\|_{L^1_t(\dot{B}^0_{2,1})} \le C_0 \quad \text{and}\\ &\|a_n\|_{\widetilde{L}^\infty_t(\dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1})} \le C_0,\end{aligned} \tag{4.2}$$

for $t < T_n^*$. Without loss of generality, we may assume T_n^* is the lifespan of the approximate solutions $(\rho_n, u_n, \nabla\Pi_n)$. Then, due to the uniform bounds (4.2), we conclude that $T_n^* \ge T_1$ for some positive constant T_1. With (4.2), we get, by using a standard compactness argument (similar to the ones in [12] for the system (1.2) in critical spaces with small initial density a_0), that (1.2) has a solution $(\rho, u, \nabla\Pi)$ so that

$$\begin{aligned}&a \in C([0,T_1];\, \dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}(\mathbb{R}^2)), \quad u \in C([0,T_1];\, \dot{B}^0_{2,1}(\mathbb{R}^2)) \cap L^1([0,T_1[;\, \dot{B}^2_{2,1}(\mathbb{R}^2)),\\ &\partial_t u, \nabla\Pi \in L^1([0,T_1]; \dot{B}^0_{2,1}(\mathbb{R}^2)).\end{aligned} \tag{4.3}$$

Furthermore, we can find some $t_0 \in (0, T_1)$ such that $u(t_0) \in H^1(\mathbb{R}^2)$. Based on the initial data $a(t_0) \in \dot{B}^{\varepsilon}_{\frac{2}{\varepsilon},1}(\mathbb{R}^2)$ and $u(t_0) \in H^1(\mathbb{R}^2)$, we may deduce the global existence of the solution to (1.2) according to [15, 22]. This completes the proof of the existence of the global solution to (1.2).

Step 2. Uniqueness of strong solutions.

Let's first recall from (2.31) that for any $t > 0$

$$\|a\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})} \le \|a_0\|_{\dot{B}^1_{2,1}} e^{C\|u\|_{L^1_t(\dot{B}^2_{2,1})}}. \tag{4.4}$$

Let $(\rho^i, u^i, \nabla\Pi^i)$ with $i = 1, 2$ be two solutions of (1.2) which satisfies (1.6), (4.3), (4.4) with $\rho = \frac{1}{1+a}$.

We set

$$(\delta a,\delta u,\nabla\delta\Pi)\stackrel{\rm def}{=}(a^2-a^1,u^2-u^1,\nabla\Pi^2-\nabla\Pi^1).$$

Then the system for $(\delta a,\delta u,\nabla\delta\Pi)$ reads

$$\left\{\begin{array}{l}\partial_t\delta a+u^2\cdot\nabla\delta a=-\delta u\cdot\nabla a^1\\ \partial_t\delta u+(u^2\cdot\nabla)\delta u-(1+a^2)(\Delta\delta u-\nabla\delta\Pi)=\delta F,\\ \mathrm{div}\,\delta u=0,\\ (\delta a,\delta u)|_{t=0}=(0,0),\end{array}\right.\tag{4.5}$$

where δF is determined by $\delta F=-(\delta u\cdot\nabla)u^1+\delta a(\Delta u^1-\nabla\Pi^1)$.

For δu, we first write the momentum equation of (4.5) as

$$\partial_t\delta u+(u^2\cdot\nabla)\delta u-(1+S_ka^2)(\Delta\delta u-\nabla\delta\Pi)=H\tag{4.6}$$

with

$$H=(a^2-S_ka^2)(\Delta\delta u-\nabla\delta\Pi)-\delta u\cdot\nabla u^1+\delta a(\Delta u^1-\nabla\Pi^1).$$

Applying Proposition 2.4 to (4.6) yields that for $\forall\,0<t\le T$

$$\begin{aligned}\|\delta\,u\|_{L^\infty_t(B^{-1}_{2,\infty})}+\|\delta\,u\|_{\widetilde L^1_t(B^1_{2,\infty})}\le{}& Ce^{C\left(t+t\|\nabla\,S_ka^2\|^2_{\widetilde L^\infty_t(B^1_{2,1})}+\|u^2\|_{L^1_t(B^1_{\infty,1})}\right)}\\ &\times\Big\{\|H\|_{\widetilde L^1_t(B^{-1}_{2,\infty})}+\|S_ka^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})}\|\nabla\delta\Pi\|_{\widetilde L^1_t(B^{-1}_{2,\infty})}\Big\}.\end{aligned}\tag{4.7}$$

Notice that

$$\begin{aligned}\|\nabla\,S_ka^2\|_{\widetilde L^\infty_t(B^1_{2,1})}&\lesssim\|\nabla\,S_ka^2\|_{\widetilde L^\infty_t(H^0)}+\|\nabla\,S_ka^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})}\lesssim\|\nabla\,a^2\|_{\widetilde L^\infty_t(H^0)}+2^k\|a^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})}\\&\lesssim\|a^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})}+2^k\|a^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})}\lesssim2^k\|a^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})}.\end{aligned}$$

which, along with (4.3), (4.4), and (4.7), follows

$$\|\delta u\|_{L^\infty_t(B^{-1}_{2,\infty})}+\|\delta u\|_{\widetilde L^1_t(B^1_{2,\infty})}\le Ce^{Ct2^k}\big(\|\nabla\delta\Pi\|_{\widetilde L^1_t(B^{-1}_{2,\infty})}+\|H\|_{\widetilde L^1_t(B^{-1}_{2,\infty})}\big).\tag{4.8}$$

On the other hand, applying div to the momentum equation of (4.5) yields

$$\mathrm{div}[(1+a^2)\nabla\delta\Pi]=\mathrm{div}\,G\tag{4.9}$$

with

$$\begin{aligned}G=&a^2\Delta\delta u-\delta u\cdot\nabla u^1-u^2\cdot\nabla\delta u+\delta a(\Delta u^1-\nabla\Pi^1)\\=&(a^2-S_ma^2)\Delta\delta u+S_ma^2\Delta\delta u-\delta u\cdot\nabla u^1-u^2\cdot\nabla\delta u+\delta a(\Delta u^1-\nabla\Pi^1)\stackrel{\rm def}{=}\sum_{\ell=1}^5\mathrm{I}_\ell.\end{aligned}$$

Thanks to Propositions 3.2 and 2.3, we get that, for any small constant $c_0>0$, there exist sufficiently large $j_0\in\mathbb{N}$ and a positive existence time T_1 such that $\|a^2-S_ja^2\|_{\widetilde L^\infty_t(B^1_{2,1})}=\|a^2-\dot S_ja^2\|_{\widetilde L^\infty_t(B^1_{2,1})}\lesssim\|a^2-\dot S_ja^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})}<c_0$, for any $j\ge j_0$ and $t\in[0,T_1]$. Then applying Proposition 2.5 to (4.9) leads to

$$\|\nabla\delta\Pi\|_{\widetilde L^1_t(B^{-1}_{2,\infty})}\lesssim\big(1+2^j\|a^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})}(1+\|a^2\|_{\widetilde L^\infty_t(\dot B^1_{2,1})})\big)\big(\|G\|_{\widetilde L^1_t(B^{-2}_{2,\infty})}+\|\mathrm{div}\,G\|_{\widetilde L^1_t(B^{-2}_{2,\infty})}\big).\tag{4.10}$$

While by Lemma 2.1 and product laws in Besov spaces in Proposition (2.2), one can see

$$\|\mathrm{I}_1\|_{\widetilde L^1_t(B^{-2}_{2,\infty})}+\|\mathrm{div}\,\mathrm{I}_1\|_{\widetilde L^1_t(B^{-2}_{2,\infty})}\lesssim\|\mathrm{I}_1\|_{\widetilde L^1_t(B^{-1}_{2,\infty})}\lesssim\|a^2-S_ma^2\|_{\widetilde L^\infty_t(B^1_{2,1})}\|\delta u\|_{\widetilde L^1_t(B^1_{2,\infty})},$$

$$\begin{aligned}\|\mathrm{I}_2\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|\mathrm{div}\,\mathrm{I}_2\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}&\lesssim\|T_{S_ma^2}\Delta\delta u\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|T_{\Delta\delta u}S_ma^2\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}\\&+\|R(S_ma^2,\Delta\delta u)\|_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}+\|T_{\nabla S_ma^2}\Delta\delta u\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|T_{\Delta\delta u}\nabla S_ma^2\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}\\&\lesssim 2^m\|\delta u\|_{\widetilde{L}^1_t(B^0_{2,\infty})}.\end{aligned}$$

Similarly, one has

$$\begin{aligned}&\|(\mathrm{I}_3,\mathrm{I}_4)\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|\mathrm{div}\,(\mathrm{I}_3,\mathrm{I}_4)\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}\\&\lesssim\|\mathrm{I}_3\|_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}+\|T_{u^2}\nabla\delta u\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|T_{\nabla\delta u}u^2\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|R(u^2_i,\partial_i\delta u)\|_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}\\&\quad+\|T_{\nabla u^2}\nabla\delta u\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|T_{\nabla\delta u}\nabla u^2\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|R(\partial_\ell u^2_i,\partial_i\delta u_\ell)\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}\\&\lesssim\int_0^t\|\delta u\|_{B^{-1}_{2,\infty}}\big(\|u^1\|_{B^1_{\infty,1}}+\|u^2\|_{B^1_{\infty,1}}\big)d\tau\end{aligned}$$

and

$$\|\mathrm{I}_5\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|\mathrm{div}\,\mathrm{I}_5\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}\lesssim\int_0^t\|\delta a\|_{L^2}\big(\|\Delta u^1\|_{L^2}+\|\nabla\Pi^1\|_{L^2}\big)d\tau.$$

Thus, we obtain

$$\begin{aligned}&\|G\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}+\|\mathrm{div}\,G\|_{\widetilde{L}^1_t(B^{-2}_{2,\infty})}\lesssim\|a^2-S_ma^2\|_{\widetilde{L}^\infty_t(B^1_{2,1})}\|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})}+2^m\|\delta u\|_{\widetilde{L}^1_t(B^0_{2,\infty})}\\&+\int_0^t\Big(\|\delta u\|_{B^{-1}_{2,\infty}}(\|u^1\|_{B^1_{\infty,1}}+\|u^2\|_{B^1_{\infty,1}})+\|\delta a\|_{L^2}(\|\Delta u^1\|_{L^2}+\|\nabla\Pi^1\|_{L^2})\Big)\,d\tau.\end{aligned}$$

which follows from (4.10) that

$$\begin{aligned}&\|\nabla\delta\Pi\|_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}\lesssim\Big\{1+2^j\|a^2\|_{\widetilde{L}^\infty_t(\dot B^1_{2,1})}(1+\|a^2\|_{\widetilde{L}^\infty_t(\dot B^1_{2,1})})\Big\}\\&\times\Big\{\|a^2-S_ma^2\|_{\widetilde{L}^\infty_t(B^1_{2,1})}\|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})}+2^m\|\delta u\|_{\widetilde{L}^1_t(B^0_{2,\infty})}\\&+\int_0^t\Big(\|\delta u\|_{B^{-1}_{2,\infty}}(\|u^1\|_{B^1_{\infty,1}}+\|u^2\|_{B^1_{\infty,1}})+\|\delta a\|_{L^2}(\|\Delta u^1\|_{L^2}+\|\nabla\Pi^1\|_{L^2})\Big)\,d\tau\Big\}.\end{aligned}\tag{4.11}$$

Toward the estimate of $\|H\|_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}$, thanks to product laws in Besov spaces in Proposition 2.2, we find

$$\begin{aligned}\|H\|_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}&\lesssim\|a^2-S_ka^2\|_{\widetilde{L}^\infty_t(B^1_{2,1})}\big(\|\Delta\delta u\|_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}+\|\nabla\delta\Pi\|_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}\big)\\&\quad+\int_0^t\|\delta u\|_{B^{-1}_{2,\infty}}\|u^1\|_{B^1_{\infty,1}}d\tau+\int_0^t\|\delta a\|_{L^2}\big(\|\Delta u^1\|_{L^2}+\|\nabla\Pi^1\|_{L^2}\big)\,d\tau.\end{aligned}\tag{4.12}$$

Taking k_0 sufficiently large and $0<T_2(\le T_1)$ small enough, one may achieve, due to (4.4), that, for any $k\ge k_0$ and $t\in(0,T_2]$

$$\|a^2-S_ka^2\|_{\widetilde{L}^\infty_t(B^1_{2,1})}\le c_0,\tag{4.13}$$

Therefore, thanks to (4.8), and (4.11)-(4.13), we prove

$$\begin{aligned}&\|\delta u\|_{L^\infty_t(B^{-1}_{2,\infty})}+\|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})}\lesssim\Big\{1+2^j\|a^2\|_{\widetilde{L}^\infty_t(\dot B^1_{2,1})}(1+\|a^2\|_{\widetilde{L}^\infty_t(\dot B^1_{2,1})})\Big\}\\&\times\Big\{\|a^2-S_ma^2\|_{\widetilde{L}^\infty_t(B^1_{2,1})}\|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})}+2^m\|\delta u\|_{\widetilde{L}^1_t(B^0_{2,\infty})}+\int_0^t\|\delta u\|_{B^{-1}_{2,\infty}}\big(\|u^1\|_{B^1_{\infty,1}}+\|u^2\|_{B^1_{\infty,1}}\big)d\tau\\&\quad+\int_0^t\|\delta a\|_{L^2}\big(\|\Delta u^1\|_{L^2}+\|\nabla\Pi^1\|_{L^2}\big)d\tau\Big\}.\end{aligned}$$

Taking m_0 sufficiently large and the positive time $T_3(\leq T_2)$ small enough, we obtain that, for any $m \geq m_0$ and $t \in (0, T_3]$

$$\begin{aligned}\|\delta u\|_{L^\infty_t(B^{-1}_{2,\infty})} + \|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})} \lesssim \Big\{1 + 2^j \|a^2\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})}(1 + \|a^2\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})})\Big\}\Big\{2^m \|\delta u\|_{\widetilde{L}^1_t(B^0_{2,\infty})} \\ + \int_0^t \|\delta u\|_{B^{-1}_{2,\infty}}\big(\|u^1\|_{B^1_{\infty,1}} + \|u^2\|_{B^1_{\infty,1}}\big)d\tau + \int_0^t \|\delta a\|_{L^2}\big(\|\Delta u^1\|_{L^2} + \|\nabla \Pi^1\|_{L^2}\big)d\tau\Big\}.\end{aligned}$$

On the other hand, by a classical estimate of the transport equation, we get from the first equation in (4.5) that

$$\|\delta a(\tau)\|_{L^2} \leq \int_0^\tau \|(\delta u \cdot \nabla)a_1\|_{L^2}\, ds \lesssim \int_0^\tau \|\delta u\|_{L^\infty}\|\nabla\, a_1\|_{L^2}\, ds \lesssim \int_0^\tau \|\delta u\|_{L^\infty}\, ds. \tag{4.14}$$

While thanks to the interpolation inequality (2.7), $\|\delta u\|_{\widetilde{L}^1_t(B^0_{2,\infty})} \lesssim \|\delta u\|^{\frac12}_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}\|\delta u\|^{\frac12}_{\widetilde{L}^1_t(B^1_{2,\infty})}$, one may prove that

$$\begin{aligned}&2^m\big(1 + 2^j\|a^2\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})}(1 + \|a^2\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})})\big)\|\delta u\|_{\widetilde{L}^1_t(B^0_{2,\infty})} \\ &\lesssim 2^m\big(1 + 2^j\|a^2\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})}(1 + \|a^2\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})})\big)\|\delta u\|^{\frac12}_{\widetilde{L}^1_t(B^{-1}_{2,\infty})}\|\delta u\|^{\frac12}_{\widetilde{L}^1_t(B^1_{2,\infty})} \\ &\leq \eta\|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})} + c_\eta 2^{2m}\big(1 + 2^{2j}\|a^2\|^2_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})}(1 + \|a^2\|_{\widetilde{L}^\infty_t(\dot{B}^1_{2,1})})^2\big)\int_0^t \|\delta u\|_{B^{-1}_{2,\infty}}\, d\tau\end{aligned}$$

As a result, we get

$$\begin{aligned}&\|\delta u\|_{L^\infty_t(B^{-1}_{2,\infty})} + \|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})} \\ &\leq C(a_0, k, j, m, \eta)\big\{\int_0^t \|\delta u\|_{B^{-1}_{2,\infty}}\big(1 + \|u^1\|_{B^1_{\infty,1}} + \|u^2\|_{B^1_{\infty,1}}\big)d\tau \\ &\qquad + \int_0^t \|\delta u\|_{L^1_\tau(L^\infty)}\big(\|\Delta u^1\|_{L^2} + \|\nabla\Pi^1\|_{L^2}\big)d\tau\big\},\end{aligned}$$

then for $T_4 \in (0, T_3]$ small enough, we obtain, $\forall\, t \in [0, T_4]$,

$$\|\delta u\|_{L^\infty_t(B^{-1}_{2,\infty})} + \|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})} \lesssim \int_0^t \|\delta u\|_{L^1_\tau(L^\infty)}\big(\|\Delta u^1\|_{L^2} + \|\nabla\Pi^1\|_{L^2}\big)d\tau. \tag{4.15}$$

Let N be an arbitrary positive integer which will be determined later on, then

$$\|\delta u\|_{L^1_\tau(L^\infty)} \leq \|\delta u\|_{L^1_\tau(\dot{B}^0_{\infty,1})} \leq \Big(\sum_{q\leq -N} + \sum_{1-N\leq q\leq N} + \sum_{q\geq N+1}\Big)\|\dot{\Delta}_q \delta u\|_{L^1_\tau(L^\infty)}.$$

Hence, due to Bernstein's inequality (Lemma 2.1), we infer

$$\begin{aligned}\|\delta u\|_{L^1_\tau(L^\infty)} &\lesssim 2^{-N}\|\delta u\|_{L^1_\tau(L^2)} + N\|\delta u\|_{\widetilde{L}^1_\tau(\dot{B}^1_{2,\infty})} + 2^{-N}\|\nabla\delta u\|_{L^1_\tau(L^\infty)} \\ &\lesssim 2^{-N}\|\delta u\|_{L^1_\tau(L^2)} + N\|\delta u\|_{\widetilde{L}^1_\tau(B^1_{2,\infty})} + 2^{-N}\|\nabla\delta u\|_{L^1_\tau(L^\infty)}\end{aligned}$$

If we choose N such that

$$N \sim \ln\big(e + \frac{\|\delta u\|_{L^1_\tau(L^2)} + \|\nabla\delta u\|_{L^1_\tau(L^\infty)}}{\|\delta u\|_{\widetilde{L}^1_\tau(B^1_{2,\infty})}}\big),$$

then there holds

$$\|\delta u\|_{L^1_\tau(L^\infty)} \lesssim \|\delta u\|_{\widetilde{L}^1_\tau(B^1_{2,\infty})}\ln\big(e + \frac{\|\delta u\|_{L^1_\tau(L^2)} + \|\nabla\delta u\|_{L^1_\tau(L^\infty)}}{\|\delta u\|_{\widetilde{L}^1_\tau(B^1_{2,\infty})}}\big),$$

and then

$$\|\delta u\|_{L^1_\tau(L^\infty)} \lesssim \|\delta u\|_{\widetilde{L}^1_\tau(B^1_{2,\infty})} \ln\big(e + \frac{\sum_{i=1}^2 \{\tau \|u^i\|_{L^\infty_\tau(L^2)} + \|\nabla u^i\|_{L^1_\tau(L^\infty)}\}}{\|\delta u\|_{\widetilde{L}^1_\tau(B^1_{2,\infty})}}\big). \tag{4.16}$$

Notice that for $\alpha \geq 0$ and $x \in (0, 1]$, there holds

$$\ln(e + \alpha x^{-1}) \leq \ln(e + \alpha)(1 - \ln x).$$

Thus, plugging (4.16) into (4.15) leads to

$$\begin{aligned} &\|\delta u\|_{L^\infty_t(B^{-1}_{2,\infty})} + \|\delta u\|_{\widetilde{L}^1_t(B^1_{2,\infty})} \\ &\lesssim \int_0^t \|\delta u\|_{\widetilde{L}^1_\tau(B^1_{2,\infty})}\big(1 - \ln \|\delta u\|_{\widetilde{L}^1_\tau(B^1_{2,\infty})}\big)\big(\|\Delta u^1\|_{L^2} + \|\nabla \Pi^1\|_{L^2}\big) d\tau. \end{aligned} \tag{4.17}$$

As $\int_0^1 \frac{dx}{x(1-\ln x)} = +\infty$, and $\|\Delta u^1\|_{L^2} + \|\nabla \Pi^1\|_{L^2}$ is locally integral in $\mathbb{R}^+$, then by Osgood's lemma (Lemma 2.4), we obtain that $\delta u(t) = 0$, which together with (4.14) and (4.11) implies that $\delta a(t) = \delta \nabla \Pi(t) = 0$ for all $t \in [0, T]$ with T small. Applying an inductive argument implies that $\delta u(t) = \delta a(t) = \delta \nabla \Pi(t) = 0$ for all $t > 0$.

Furthermore, applying (4.17) (up to a slight modification) to the system (1.1), we may readily prove that the solution $(a,\, u) \in C(\mathbb{R}_+;\, \dot{B}^1_{2,1}(\mathbb{R}^2)) \times C(\mathbb{R}_+;\, \dot{B}^0_{2,1}(\mathbb{R}^2))$ depends continuously on the initial data $(a_0,\, u_0) \in \dot{B}^1_{2,1}(\mathbb{R}^2) \times \dot{B}^0_{2,1}(\mathbb{R}^2)$. This completes the proof of Theorem 1.1. □

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